The Probably Approximately Correct (PAC) Learning

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UMB
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What is the PAC Model?

Definition

A hypothesis class $\mathcal{H}$ is PAC learnable if there exists a function $m_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm $A$ such that for every $\epsilon, \delta \in (0, 1)$, every distribution $\mathcal{D}$ over $\mathcal{X}$, and for every labeling function $f : \mathcal{X} \rightarrow \{0, 1\}$, if realizable assumption holds with respect to $\mathcal{H}, \mathcal{D}, f$, then when running the algorithm on $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ iid examples generated by $\mathcal{D}$ and labeled by $f$, $A$ returns a hypothesis $h$ such that, with probability at least $1 - \delta$ (over the choice of examples), we have $L_{(\mathcal{D}, f)}(h) \leq \epsilon$. 
Approximation Parameters

- The **accuracy parameter** $\epsilon$ determines how far the output classifier can be from the optimal one, and
- The **confidence parameter** $\delta$ indicates how likely is the classifier is to meet that accuracy requirement.
What is Agnostic PAC Learning

- The realizability assumption (the existence of a hypothesis \( h^* \in \mathcal{H} \) such that \( P_{x \sim D}(h^*(x) = f(x)) = 1 \)) is not realistic in many cases.
- Agnostic learning replaces the realizability assumption and the targeted labeling function \( f \), with a distribution \( D \) defined on pairs (data, labels), that is with a distribution \( D \) on \( \mathcal{X} \times \mathcal{Y} \).
When the probability distribution $\mathcal{D}$ was defined on $\mathcal{X}$, the generalization error of a hypothesis was defined as:

$$L_{\mathcal{D},f}(h) = \mathcal{D}(\{x \mid h(x) \neq f(x)\}).$$

Now, $\mathcal{D}$ is defined over $\mathcal{X} \times \mathcal{Y}$, so we redefine the generalization error as:

$$L_{\mathcal{D}}(h) = \mathcal{D}(\{(x,y) \mid h(x) \neq y\}).$$

We seek a predictor for which $L_{\mathcal{D}}(h)$ is minimal.

The definition of the empirical risk remains the same:

$$L_{S}(h) = \frac{|\{i \mid h(x_i) \neq y_i \text{ for } 1 \leq i \leq m\}|}{m}.$$
Let $\mathcal{D}$ be any probability distribution over $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{Y} = \{0, 1\}$.

Let $X$ be a random variable ranging over $\mathcal{X}$ and $Y$ be a random variable ranging over $\mathcal{Y} = \{0, 1\}$.

The Bayes predictor is the function $f_{\mathcal{D}}$ defined as

$$ f_{\mathcal{D}}(x) = \begin{cases} 1 & \text{if } P(Y = 1|X = x) \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} $$
Theorem

Given any probability distribution $\mathcal{D}$ over $\mathcal{X} \times \{0, 1\}$ the best label predicting function from $\mathcal{X}$ to $\{0, 1\}$ is the Bayes predictor.

Proof: Let $X$ be a random variable ranging over $\mathcal{X}$, $Y$ be a random variable ranging over $\mathcal{Y} = \{0, 1\}$, and let $\alpha_x$ be the probability of a having a label 1 given $x$, that is,

$$\alpha_x = P(Y = 1 | X = x).$$

In other words, the Bayes predictor is

$$f_D(x) = \begin{cases} 1 & \text{if } \alpha_x \geq \frac{1}{2} \\ 0 & \text{if } \alpha_x < \frac{1}{2}. \end{cases}$$
We have:

\[ L_D(f_D) = P(f_D(X) \neq y | X = x) \]
\[ = P(f_D(x) = 1 | X = x)P(Y = 0 | X = x) + P(f_D(x) = 0 | X = x)P(Y = 1 | X = x) \]
\[ = P\left(\alpha x \geq \frac{1}{2}\right)P(Y = 0 | X = x) + P\left(\alpha x < \frac{1}{2}\right)P(Y = 1 | X = x) \]
Proof (cont’d)

If $\alpha_x \geq \frac{1}{2}$, then $\min\{\alpha_x, 1 - \alpha_x\} = 1 - \alpha_x$, $P\left(\alpha_x \geq \frac{1}{2}\right) = 1$, $P\left(\alpha_x < \frac{1}{2}\right) = 0$ and

$$P\left(\alpha_x \geq \frac{1}{2}\right) \left(1 - \alpha_x\right) + P\left(\alpha_x < \frac{1}{2}\right) \alpha_x = 1 - \alpha_x = \min\{1 - \alpha_x, \alpha_x\}.$$ 

If $\alpha_x < \frac{1}{2}$, then $\min\{\alpha_x, 1 - \alpha_x\} = \alpha_x$, $P\left(\alpha_x \geq \frac{1}{2}\right) = 0$, $P\left(\alpha_x < \frac{1}{2}\right) = 1$ and

$$P\left(\alpha_x \geq \frac{1}{2}\right) \left(1 - \alpha_x\right) + P\left(\alpha_x < \frac{1}{2}\right) \alpha_x = \alpha_x = \min\{1 - \alpha_x, \alpha_x\}.$$
Let $g$ be any other classifier. We have:

$$P(g(X) \neq Y | X = x) = P(g(X) = 0 | X = x)P(Y = 1 | X = x) + P(g(X) = 1 | X = x)P(Y = 0 | X = x)$$

$$= P(g(X) = 0 | X = x)\alpha_x + P(g(X) = 1 | X = x)(1 - \alpha_x)$$

$$\geq P(g(X) = 0 | X = x) \min\{\alpha_x, 1 - \alpha_x\} + P(g(X) = 1 | X = x) \min\{\alpha_x, 1 - \alpha_x\}$$

$$\cdot \min\{\alpha_x, 1 - \alpha_x\}$$

$$= \min\{\alpha_x, 1 - \alpha_x\} = P(f_D(X) \neq y | X = x).$$
A hypothesis class $\mathcal{H}$ is agnostic PAC learnable if there exists a function $m_\mathcal{H} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm $A$ with the following property: For every $\epsilon, \delta \in (0, 1)$ and for every distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, when running $A$ on $m \geq m_\mathcal{H}(\epsilon, \delta)$ iid examples generated by $\mathcal{D}$, $A$ returns a hypothesis $h$ such that with probability at least $1 - \delta$ (over the choice of the $m$ training examples) we have

$$L_\mathcal{D}(h) \leq \min_{h' \in \mathcal{H}} L_\mathcal{D}(h') + \epsilon.$$
If the realizability assumption holds, agnostic PAC learning provides the same guarantees as PAC learning.

When the realizability assumption does not hold, no learner can guarantee an arbitrary small error.

A learner $\mathcal{A}$ can declare success if the error is not much larger than the smallest error achievable by a hypothesis from $\mathcal{H}$. 
Multiclass Classification

Example
Let $\mathcal{X}$ be a set of document features, and $\mathcal{Y}$ a set of topics (sports, politics, health, etc.). By document features we mean counts of certain key words, size, or origin of the document. The loss function will be the probability of the event that occurs when the predictor suggest a wrong label.
The Scope of Learning Problems

Example

In regression we seek to find a functional relationship $h$ between the $\mathcal{X}$ and $\mathcal{Y}$ components of the data. For example, to predict the weight of a baby at birth $\mathcal{X}$ can be a set of triplets in $\mathbb{R}^3$ (head circumference, abdominal circumference, femur length) and $\mathcal{Y}$ is is the weight at birth. We seek $h$ that will minimize the loss

$$L_D(h) = E_{(x,y) \sim D}(h(x) - y)^2.$$
Generalized Loss Functions

Definition
Given a set of hypotheses $\mathcal{H}$, a domain $Z$, a loss function is a function $\ell : \mathcal{H} \times Z \rightarrow \mathbb{R}_+$. For prediction problems we have $Z = \mathcal{X} \times \mathcal{Y}$.

Definition
The risk function is the expected loss of the classifier $h \in \mathcal{H}$ with respect to a probability distribution $\mathcal{D}$ over $Z$, namely

$$L_{\mathcal{D}}(h) = E_{z \sim \mathcal{D}}(\ell(h, z)).$$

The empirical risk is the expected loss over the sample $S = (z_1, \ldots, s_m) \in Z^m$ as

$$L_S(h) = \frac{1}{m} \sum_{i=1}^{m} \ell(h, z_i).$$
The random variable $z$ ranges over $X \times Y$ and the loss function is

$$
\ell_{0-1}(h, (x, y)) = \begin{cases} 
0 & \text{if } h(x) = y, \\
1 & \text{if } h(x) \neq y.
\end{cases}
$$

This is used in binary or multiclass classification problems. For the 0/1 loss the definition of $L_D(h) = E_{z \sim D}(\ell(h, z))$ coincides with the previous definition in the agnostic PAC, $L_D(h) = D(\{(x, y) \mid h(x) \neq y\})$. 
The random variable $z$ ranges over $\mathcal{X} \times \mathcal{Y}$ and the loss function is

$$\ell_{sq}(h, (x, y)) = (h(x) - y)^2.$$
Definition

A hypothesis class $\mathcal{H}$ is **agnostic PAC learnable** with respect to a set $Z$ and a loss function $\ell : \mathcal{H} \times Z \rightarrow \mathbb{R}_+$ if there exists a function $m_\mathcal{H} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm $A$ with the following property: For every $\epsilon, \delta \in (0, 1)$ and for every distribution $D$ over $Z$, when running $A$ on $m \geq m_\mathcal{H}(\epsilon, \delta)$ iid examples generated by $D$, $A$ returns a hypothesis $h$ such that with probability at least $1 - \delta$ (over the choice of the $m$ training examples) we have

$$L_D(h) \leq \min_{h' \in \mathcal{H}} L_D(h') + \epsilon,$$

where $L_D(h) = E_{z \sim D}(\ell(h, z))$. 