CS724: Topics in Algorithms
Eigenvalues of Matrices

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Definition

Let \( A \in \mathbb{C}^{m \times m} \) be a square matrix. A pair \((\lambda, x)\) that consists of a complex number \( \lambda \in \mathbb{C} \) and a non-zero complex vector \( x \) is an eigenpair if \( Ax = \lambda x \). The number \( \lambda \) is an eigenvalue of \( A \), and \( x \) is an eigenvector of \( A \). The set of eigenvalues of \( A \) is known as the spectrum of \( A \) and is denoted by \( \text{spec}(A) \). The spectral radius of \( A \) is the number

\[
\rho(A) = \max\{|\lambda| \mid \lambda \in \text{spec}(A)\}.
\]
Theorem

Let $A \in \mathbb{C}^{n \times n}$ be a matrix and let $\lambda_1, \ldots, \lambda_n$ distinct eigenvalues of $A$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors of $A$ that correspond to distinct eigenvalues, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.
Proof

If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ were not linearly independent we would have a linear combination of these vectors

$$c_{i_1} \mathbf{v}_{i_1} + \cdots + c_{i_p} \mathbf{v}_{i_p} = \mathbf{0}_n,$$

containing a minimal number of vectors such that not every one of scalars $c_{i_1}, \ldots, c_{i_p}$ is 0. This implies

$$c_{i_1} A \mathbf{v}_{i_1} + \cdots + c_{i_p} A \mathbf{v}_{i_p} = c_{i_1} \lambda_{i_1} \mathbf{v}_{i_1} + \cdots + c_{i_p} \lambda_{i_p} \mathbf{v}_{i_p} = \mathbf{0}.$$

These equalities imply

$$c_{i_1} (\lambda_{i_1} - \lambda_{i_p}) \mathbf{v}_{i_1} + \cdots + c_{i_{p-1}} (\lambda_{i_{p-1}} - \lambda_{i_p}) \mathbf{v}_{i_{p-1}} = \mathbf{0}_n,$$

which contradicts the minimality of the number of terms. Thus, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.
Corollary

If $A \in \mathbb{C}^{n \times n}$ then the set of eigenvalues of $A$ does not contain more than $n$ distinct eigenvalues.

Proof.

Since the maximum size of a linearly independent set in $\mathbb{C}^n$ is $n$, it follows that $A$ cannot have more than $n$ distinct eigenvalues.
Definition

Let $A \in \mathbb{C}^{n \times n}$ be a matrix. A subspace $S$ of $\mathbb{C}^n$ is a right invariant subspace of the matrix $A$ if $As \in S$ for every $s \in S$, and is a left invariant subspace if $A^H s \in S$ for every $s \in S$. 
Definition

The *geometric multiplicity of an eigenvalue* $\lambda$ of a matrix $A \in \mathbb{C}^{n \times n}$ is denoted by $\text{geom}(A, \lambda)$ and is equal to $\dim(S_{A, \lambda})$. Equivalently, the geometric multiplicity of $\lambda$ is

$$\text{geom}(A, \lambda) = \dim(\text{NullSp}(A - \lambda I_n)) = n - \text{rank}(A - \lambda I_n).$$
Theorem

Let $A \in \mathbb{R}^{n \times n}$. We have $0 \in \text{spec}(A)$ if and only if $A$ is a singular matrix. Moreover, in this case, $\text{geom}(A, 0) = n - \text{rank}(A) = \dim(\text{NullSp}(A))$. 
Example

The matrix $I_n$ has 1 as its unique eigenvalue. Its invariant subspace is the entire space $V$; therefore, the geometric multiplicity of 1 is $\dim(V)$. 
Definition

Let \( A \in \mathbb{C}^{m \times n} \) be a matrix. An \emph{invariant subspace of} \( A \) is a subspace \( S \) of \( \mathbb{R}^n \) such that \( x \in S \) implies \( Ax \in S \).

- the null space of a matrix \( A \) is an invariant subspace;
- if \( x \) is an eigenvector of \( A \), then \( \{ax \mid a \in \mathbb{C}\} \) is an invariant subspace of \( A \).
If \( \lambda \) is an eigenvalue of \( A \in \mathbb{C}^{n \times n} \) we have \( x^H A x = \lambda x^H x \), so

\[
\lambda = \frac{x^H A x}{x^H x}.
\]

In the real case we replace \( x^H \) by \( x' \): if \( A \in \mathbb{R}^{n \times n} \), \( \lambda \) is an eigenvalue and \( x \) is an eigenvector that corresponds to \( \lambda \), then

\[
\lambda = \frac{x' A x}{x' x}.
\]
Theorem

Let $A \in \mathbb{C}^{n \times n}$ and let $S \subseteq \mathbb{C}^n$ be an invariant subspace of $A$. If the columns of a matrix $X \in \mathbb{C}^{n \times p}$ constitute a basis of $S$, then there exists a unique matrix $L \in \mathbb{C}^{p \times p}$ such that $AX = XL$. 
Proof

Let $X = (x_1 \cdots x_p)$. Since $A x_1 \in S$ it follows that $A x_1$ can be uniquely expressed as a linear combination of the columns of $X$, that is,

$$A x_j = x_1 \ell_1 j + \cdots + x_p \ell_p j$$

for $1 \leq i \leq p$. Thus,

$$A x_j = X \begin{pmatrix} \ell_1 j \\ \vdots \\ \ell_p j \end{pmatrix}.$$ 

The matrix $L$ is defined by $L = (\ell_{ij})$.

**Corollary**

$(\lambda, v)$ is an eigenpair of $L$ if and only if $(\lambda, X v)$ is an eigenpair of $A$. 
Let $A \in \mathbb{C}^{n \times n}$ be a matrix having the eigenvalues $\lambda_1, \ldots, \lambda_n$. If $x_1, \ldots, x_n$ are $n$ eigenvectors corresponding to these values, then we have

$$Ax_1 = \lambda_1 x_1, \ldots, Ax_n = \lambda_n x_n. \quad (1)$$

By introducing the matrix $X = (x_1 \cdots x_n) \in \mathbb{C}^{n \times n}$ these equalities can be written in a concentrated form as

$$AX = X \text{diag}(\lambda_1, \ldots, \lambda_n). \quad (2)$$

Obviously, since the eigenvalues can be listed in several ways, this equality is not unique.
Suppose now that $x_1, \ldots, x_n$ are unit vectors and that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. Then $X$ is a unitary matrix, $X^{-1} = X^H$ and we obtain the equality

$$A = X \text{diag}(\lambda_1, \ldots, \lambda_n) X^H = \lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H$$

known as the *spectral decomposition* of the matrix $A$. 
If $\lambda$ is an eigenvalue of the matrix $A \in \mathbb{C}^{n \times n}$, there exists a non-zero eigenvector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. Therefore, the linear system

$$(\lambda I_n - A)x = 0_n$$

has a non-trivial solution. This is possible if and only if $\det(\lambda I_n - A) = 0$, so eigenvalues are the solutions of the equation

$$\det(\lambda I_n - A) = 0.$$ 

Note that $\det(\lambda I_n - A)$ is a polynomial of degree $n$ in $\lambda$, known as the \textit{characteristic polynomial} of the matrix $A$. We denote this polynomial by $p_A$. 
Example

Let

\[ A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} \]

be a matrix in \( \mathbb{C}^{3 \times 3} \). Its characteristic polynomial is

\[ p_A = \begin{vmatrix}
  \lambda - a_{11} & -a_{12} & -a_{13} \\
  -a_{21} & \lambda - a_{22} & -a_{23} \\
  -a_{31} & -a_{32} & \lambda - a_{33}
\end{vmatrix} = \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 \\
+ (a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31})\lambda \\
- (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} - a_{13}a_{31}a_{22}) \]
Theorem

Let $A \in \mathbb{C}^{n \times n}$. Then, $\text{spec}(A) = \text{spec}(A')$ and $\text{spec}(A^H) = \{\overline{\lambda} \mid \lambda \in \text{spec}(A)\}$.

Proof: We have

$$p_{A'}(\lambda) = \det(\lambda I_n - A') = \det((\lambda I_n - A)') = \det(\lambda I_n - A) = p_A(\lambda).$$

Thus, since $A$ and $A'$ have the same characteristic polynomials, their spectra are the same.

For $A^H$ we can write

$$p_{A^H}(\overline{\lambda}) = \det(\overline{\lambda} I_n - A^H) = \det((\lambda I_n - A)^H) = (p_A(\lambda))^H,$$

which implies the second part of the Theorem.
The characteristic polynomial of a matrix can be computed in R using the function `charpoly` of the `pracma` package. For the matrix $A$ defined in

```r
> A <- matrix(c(1:6),3,3)
> A
[,1] [,2] [,3]
[1,] 1 4 1
[2,] 2 5 2
[3,] 3 6 3
```
the characteristic polynomial is $\lambda^3 - 9\lambda^2$ as returned by

```r
> charpoly(A)
[1] 1 -9 0 0
```
Let $B$ be the matrix defined as

```r
> B <- matrix(c(1,0,2,3,1,1,1,4,2),3,3)
> B
[,1] [,2] [,3]
[1,] 1 3 1
[2,] 0 1 4
[3,] 2 1 2
```
If `charpoly` is called as in

```r
> charpoly(B, info=TRUE)
```

then, in addition to the characteristic polynomial of $B$, its determinant and inverse matrix are also returned as in

$\text{cp}$

```
[1] 1 -4 -1 -20
```

$\text{det}$

```
[1] 20
```

$\text{inv}$

```
[,1] [,2] [,3]
[1,] -0.1 -0.25 0.55
[2,] 0.4 0.00 -0.20
[3,] -0.1 0.25 0.05
```
To compute the eigenvalues of a matrix one could use the \texttt{eigen} function of the base package of \texttt{R}.

The following call to \texttt{eigen} computes the eigenvalues of the matrix \(A\) together with its characteristic vectors:

\begin{verbatim}
> A <- matrix(c(1:6),3,3)
> eigen(A)

\texttt{eigen()} decomposition

$\text{values}$

\[
\begin{array}{ccc}
1 & \text{values} \\
1 & 9.000000e+00 & 2.497182e-09 & -2.497182e-09
\end{array}
\]

$\text{vectors}$

\[
\begin{array}{ccc}
\text{[,1]} & \text{[,2]} & \text{[,3]} \\
[1,] & 0.3713907 & -7.071068e-01 & 7.071068e-01 \\
[2,] & 0.5570860 & -1.177183e-09 & -1.177183e-09 \\
[3,] & 0.7427814 & 7.071068e-01 & -7.071068e-01
\end{array}
\]
Equality of spectra of $A$ and $A'$ does not imply that the eigenvectors or the invariant subspaces of the corresponding eigenvalues are identical, as it can be seen from the following example.
Example

Consider the matrix $A \in \mathbb{C}^{2 \times 2}$ defined by

$$A = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix},$$

where $a \neq b$ and $c \neq 0$. It is immediate that $\text{spec}(A) = \text{spec}(A') = \{a, b\}$. For $\lambda_1 = a$ we have the distinct invariant subspaces:

$$S_{A,a} = \left\{ k \begin{pmatrix} a-b \\ c \end{pmatrix} \mid k \in \mathbb{C} \right\}$$

$$S_{A',a} = \left\{ k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid k \in \mathbb{C} \right\},$$

as the reader can easily verify.
The leading term of the characteristic polynomial of $A$ is generated by $(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$ and equals $\lambda^n$.

The fundamental theorem of algebra implies that $p_A$ has $n$ complex roots, not necessarily distinct. Observe also that, if $A$ is a matrix with real entries, the roots are paired as conjugate complex numbers.
Definition

The \textit{algebraic multiplicity of an eigenvalue} \( \lambda \) of a matrix \( A \in \mathbb{C}^{n \times n} \), \( \text{algm}(A, \lambda) \) equals \( k \) if \( \lambda \) is a root of order \( k \) of the equation \( p_A(\lambda) = 0 \). If \( \text{algm}(A, \lambda) = 1 \), we refer to \( \lambda \) as a \textit{simple eigenvalue}. 

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Let $A \in \mathbb{R}^{3 \times 3}$ be the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A$ is

$$p_A(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 & -2 \\ -2 & -1 & \lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - 3\lambda.$$

Therefore, the eigenvalues of $A$ are 3, 0 and $-1$. 
The eigenvalues of $I_3$ are obtained from the equation

$$det(\lambda I_3 - I_3) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3 = 0.$$ 

Thus, $I_3$ has one eigenvalue, 1, and $\text{algm}(I_3, 1) = 3$. 
Theorem

Let $A \in \mathbb{R}^{n \times n}$ be a matrix and let $\lambda \in \text{spec}(A)$. Then, for any $k \in \mathbb{P}$, $\lambda^k \in \text{spec}(A^k)$.

Proof.

The proof is by induction on $k \geq 1$. The base step, $k = 1$ is immediate. Suppose that $\lambda^k \in \text{spec}(A^k)$, that is $A^k \mathbf{x} = \lambda^k \mathbf{x}$ for some $\mathbf{x} \in V - \{0\}$. Then, $A^{k+1} \mathbf{x} = A(A^k \mathbf{x}) = A(\lambda^k \mathbf{x}) = \lambda^k A \mathbf{x} = \lambda^{k+1} \mathbf{x}$, so $\lambda^{k+1} \in \text{spec}(A^{k+1})$. \qed
**Theorem**

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $\lambda \in \text{spec}(A)$. We have $\frac{1}{\lambda} \in \text{spec}(A^{-1})$ and the sets of eigenvectors of $A$ and $A^{-1}$ are equal.

**Proof.**

Since $\lambda \in \text{spec}(A)$ and $A$ is non-singular we have $\lambda \neq 0$ and $A\mathbf{x} = \lambda \mathbf{x}$ for some $\mathbf{x} \in V - \{\mathbf{0}\}$. Therefore, we have $A^{-1}(A\mathbf{x}) = \lambda A^{-1}\mathbf{x}$, which is equivalent to $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which implies $\frac{1}{\lambda} \in \text{spec}(A^{-1})$. In addition, this implies that the set of eigenvectors of $A$ and $A^{-1}$ are identical. \qed
Theorem

Let \( p_A(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n \) be the characteristic polynomial of the matrix \( A \). Then, we have \( c_i = (-1)^i S_i(A) \) for \( 1 \leq i \leq n \), where \( S_i(A) \) is the sum of all principal minors of order \( i \) of \( A \).
Proof

Since \( p_A(\lambda) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n \), it is easy to see that the derivatives of \( p_A(\lambda) \) are given by:

\[
p^{(1)}_A(\lambda) = n\lambda^{n-1} + (n - 1)c_1 \lambda^{n-2} + \cdots + c_{n-1},
\]

\[
p^{(2)}_A(\lambda) = n(n - 1)\lambda^{n-2} + (n - 1)(n - 2)c_1 \lambda^{n-3} + \cdots + 2c_{n-2}),
\]

\[
\vdots
\]

\[
p^{(k)}_A(\lambda) = n(n - 1)\cdots(n - k + 1)\lambda^{n-k} + \cdots + k!c_{n-k}),
\]

\[
\vdots
\]

\[
p^{(n)}_A(\lambda) = n!c_0.
\]

This implies

\[ c_{n-k} = k!p^{(k)}_A(0) \]

for \( 0 \leq k \leq n \).

On other hand, \( c_{n-k} = \frac{1}{k!}(-1)^k k! S_{n-k}(A) = (-1)^{n-k} S_{n-k}(A) \), which implies the statement of theorem.
By Viète’s Theorem, taking into account Theorem 17 we have:

\[ \lambda_1 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = \text{trace}(A) = -c_1. \]

Another interesting fact is

\[ \lambda_1 \cdots \lambda_n = \det(A). \]
Theorem

Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$ be two matrices. Then the set of non-zero eigenvalues of the matrices $AB \in \mathbb{C}^{m \times m}$ and $BA \in \mathbb{C}^{n \times n}$ are the same and $\text{alg}_m(AB, \lambda) = \text{alg}_m(BA, \lambda)$ for each such eigenvalue.
Proof

Consider the following straightforward equalities:

\[
\begin{pmatrix}
I_m & -A \\
O_{n,m} & \lambda I_n
\end{pmatrix}
\begin{pmatrix}
\lambda I_m & A \\
B & I_n
\end{pmatrix}
= 
\begin{pmatrix}
\lambda I_m - AB & O_{m,n} \\
-\lambda B & \lambda I_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
-I_m & O_{m,n} \\
-B & \lambda I_n
\end{pmatrix}
\begin{pmatrix}
\lambda I_m & A \\
B & I_n
\end{pmatrix}
= 
\begin{pmatrix}
-\lambda I_m & -A \\
O_{n,m} & \lambda I_n - BA
\end{pmatrix}.
\]

Observe that

\[
\det \begin{pmatrix}
I_m & -A \\
O_{n,m} & \lambda I_n
\end{pmatrix}
\begin{pmatrix}
\lambda I_m & A \\
B & I_n
\end{pmatrix}
= 
\det \begin{pmatrix}
-I_m & O_{m,n} \\
-B & \lambda I_n
\end{pmatrix}
\begin{pmatrix}
\lambda I_m & A \\
B & I_n
\end{pmatrix},
\]

and therefore,

\[
\det \begin{pmatrix}
\lambda I_m - AB & O_{m,n} \\
-\lambda B & \lambda I_n
\end{pmatrix}
= 
\det \begin{pmatrix}
-\lambda I_m & -A \\
O_{n,m} & \lambda I_n - BA
\end{pmatrix}.
\]

The last equality amounts to

\[
\lambda^n p_{AB}(\lambda) = \lambda^m p_{BA}(\lambda).
\]

Thus, for \( \lambda \neq 0 \) we have \( p_{AB}(\lambda) = p_{BA}(\lambda) \), which gives the desired conclusion.
Corollary

Let

\[ a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \]

be a vector in \( \mathbb{C}^n - \{0\} \). Then, the matrix \( aa^H \in \mathbb{C}^{n \times n} \) has one eigenvalue distinct from 0, and this eigenvalue is equal to \( \| a \|_2^2 \).
Theorem

Let $A \in \mathbb{C}^{(m+n)\times(m+n)}$ be a matrix partitioned as

$$A = \begin{pmatrix} B & C \\ O_{n,m} & D \end{pmatrix},$$

where $B \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}^{n\times n}$. Then, $\text{spec}(A) = \text{spec}(B) \cup \text{spec}(D)$. 
Proof

Let $\lambda \in \text{spec}(A)$ and let $x \in \mathbb{C}^{m+n}$ be an eigenvector that corresponds to $\lambda$. If

$$x = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$, then we have

$$Ax = \begin{pmatrix} B & C \\ O_{n,m} & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Bu + Cv \\ Dv \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}.$$

This implies $Bu + Cv = \lambda u$ and $Dv = \lambda v$. If $v \neq \mathbf{0}$, then $\lambda \in \text{spec}(D)$; otherwise, $Bu = \lambda u$, which yields $\lambda \in \text{spec}(B)$, so $\lambda \in \text{spec}(B) \cup \text{spec}(D)$. Thus, $\text{spec}(A) \subseteq \text{spec}(B) \cup \text{spec}(D)$. 

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To prove the converse inclusion, note that if $\lambda \in \text{spec}(B)$ and $u$ is an eigenvector of $\lambda$, then $Bu = \lambda u$, which means that

$$A \begin{pmatrix} u \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 0 \end{pmatrix},$$

so $\text{spec}(B) \subseteq \text{spec}(A)$. Similarly, $\text{spec}(D) \subseteq \text{spec}(A)$, which implies the equality of the theorem.
Theorem

All eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ are real numbers. All eigenvalues of a skew-Hermitian matrix are purely imaginary numbers.

Proof.

Note that $\mathbf{x}^H \mathbf{x}$ is a real number for every $\mathbf{x} \in \mathbb{C}^n$. Since $\lambda = \frac{\mathbf{x}^H A \mathbf{x}}{\mathbf{x}^H \mathbf{x}}$, $\lambda$ is a real number.

Suppose now that $B$ is a skew-Hermitian matrix. Then, as above, $\mathbf{x}^H A \mathbf{x} = -\mathbf{x}^H A \mathbf{x}$, which implies that the real part of $\mathbf{x}^H A \mathbf{x}$ is 0. Thus, $\mathbf{x}^H A \mathbf{x}$ is a purely imaginary number and $\lambda$ is a purely imaginary number. \qed
Corollary

If \( A \in \mathbb{R}^{n \times n} \) and \( A \) is a symmetric matrix, then all its eigenvalues are real numbers.

Proof.

This statement follows from Theorem 21 by observing that the Hermitian adjoint \( A^H \) of a matrix \( A \in \mathbb{R}^{n \times n} \) coincides with its transposed matrix \( A' \).
Corollary

Let \( A \in \mathbb{C}^{m \times n} \) be a matrix. The non-zero eigenvalues of the matrices \( AA^H \) and \( A^H A \) are positive numbers and they have the same algebraic multiplicities for the matrices \( AA^H \) and \( A^H A \).
Proof

We need to verify only that if \( \lambda \) is a non-zero eigenvalue of \( A^H A \), then \( \lambda \) is a positive number. Since \( A^H A \) is a Hermitian matrix, \( \lambda \) is a real number. The equality \( A^H A x = \lambda x \) for some eigenvector \( x \neq 0 \) implies

\[
\lambda \| x \|_2^2 = \lambda x^H x = (Ax)^H Ax = \| Ax \|_2^2,
\]

so \( \lambda > 0 \).
Corollary

Let $A \in \mathbb{C}^{m \times n}$ be a matrix. The eigenvalues of the matrix $B = A^H A \in \mathbb{C}^{n \times n}$ are real non-negative numbers.
The matrix $B$ defined above is clearly Hermitian and, therefore, its eigenvalues are real numbers. Next, if $\lambda$ is an eigenvalue of $B$, then

$$\lambda = \frac{x^H A^H A x}{x^H x} = \frac{(Ax)^H A x}{x^H x} = \frac{||Ax||}{||x||} \geq 0,$$

where $x$ is an eigenvector that corresponds to $\lambda$. 
If $A$ is a Hermitian matrix, then $A^H A = A^2$, hence the spectrum of $A^H A$ is \{ $\lambda^2$ $|$ $\lambda \in \text{spec}(A)$ \}. 

**Theorem**

*If $A \in \mathbb{C}^{n \times n}$ is a Hermitian matrix and $u, v$ are two eigenvectors that correspond to two distinct eigenvalues $\lambda_1$ and $\lambda_2$, then $u \perp v$.***
Proof.

We have $Au = \lambda_1 u$ and $Av = \lambda_2 v$. This allows us to write $v^H Au = \lambda_1 v^H u$. Since $A$ is Hermitian, we have 

$$\lambda_1 v^H u = v^H Au = v^H A^H u = (Av)^H u = \lambda_2 v^H u,$$

which implies $v^H u = 0$, that is, $u \perp v$.  

□
Theorem

If \( A, B \in \mathbb{C}^{n \times n} \) and \( A \sim B \), then the two matrices have the same characteristic polynomials and, therefore, \( \text{spec}(A) = \text{spec}(B) \).

Proof.

Since \( A \sim B \), there exists an invertible matrix \( X \) such that \( A = XBX^{-1} \). Then, the characteristic polynomial \( \det(A - \lambda I_n) \) can be rewritten as

\[
\begin{align*}
\det(A - \lambda I_n) & = \det(XBX^{-1} - \lambda XI_nX^{-1}) \\
& = \det((X(B - \lambda I_n)X^{-1}) \\
& = \det(X) \det(B - \lambda I_n) \det(X^{-1}) \\
& = \det(B - \lambda I_n),
\end{align*}
\]

which implies \( \text{spec}(A) = \text{spec}(B) \). \( \square \)
Theorem

If \( A, B \in \mathbb{C}^{n \times n} \) and \( A \sim B \), then \( \text{trace}(A) = \text{trace}(B) \).

Proof.

Since the two matrices are similar, they have the same characteristic polynomials, so both \( \text{trace}(A) \) and \( \text{trace}(B) \) equal \(-c_1\), where \( c_1 \) is the coefficient of \( \lambda^{n-1} \) in both \( p_A(\lambda) \) and \( p_B(\lambda) \).
Theorem

If $A \sim_u B$, where $A, B \in \mathbb{C}^{n \times n}$, then the Frobenius norm of these matrices are equal, that is, $\| A \|_F = \| B \|_F$.

Proof.

Since $A \sim_u B$, there exists a unitary matrix $U$ such that $A = UBU^H$. Therefore,

$$A^H A = U B^H U^H U B U^H = U B^H B U^H,$$

which implies $A^H A \sim_u B^H B$. Therefore, these matrices have the same characteristic polynomials which allows us to infer that $trace(A^H A) \sim_u trace(B^H B)$, which yields the desired equality. \qed
Theorem

Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{k \times k}$ be two matrices. If there exists a matrix $U \in \mathbb{C}^{n \times k}$ having an orthonormal set of columns such that $AU = UB$, then there exists $V \in \mathbb{C}^{n \times (n-k)}$ such that $(U \ V) \in \mathbb{C}^{n \times n}$ is a unitary matrix and

$$(U \ V)^H A (U \ V) = \begin{pmatrix} B & U^H AV \\ O & V^H AV \end{pmatrix}.$$
Proof

Since $U$ has an orthonormal set of columns, there exists $V \in \mathbb{C}^{n \times (n-k)}$ such that $(U \ V)$ is a unitary matrix.

We have

$$U^H AU = U^H UB = I_k B = B,$$
$$V^H AU = V^H UB = OB = O,$$

which allows us to write

$$(U \ V)^H A(U \ V) = (U \ V)^H (AU \ AV) = \begin{pmatrix} \ U^H \\ V^H \end{pmatrix} (AU \ AV) = \begin{pmatrix} \ U^H AU & U^H AV \\ V^H AU & V^H AV \end{pmatrix} = \begin{pmatrix} B & U^H AV \\ O & V^H AV \end{pmatrix}.$$
Corollary

Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix and $B \in \mathbb{C}^{k \times k}$ be a matrix. If there exists a matrix $U \in \mathbb{C}^{n \times k}$ having an orthonormal set of columns such that $AU = UB$, then there exists $V \in \mathbb{C}^{n \times (n-k)}$ such that $(U \ V)$ is a unitary matrix and

$$(U \ V)^H A (U \ V) = \begin{pmatrix} B & O \\ O & V^H A V \end{pmatrix}.$$ 

Proof.

Since $A$ is Hermitian we have $U^H AV = U^H A^H V = (V^H AU)^H = O$, which produces the desired result.
Corollary

Let $A \in \mathbb{C}^{n\times n}$, $\lambda$ be an eigenvalue of $A$, and let $u$ be an eigenvector of $A$ with $\|u\| = 1$ that corresponds to $\lambda$. There exists $V \in \mathbb{C}^{n\times(n-1)}$ such that $(u \ V) \in \mathbb{C}^{n\times n}$ is a unitary matrix and

$$(u \ V)^H A (u \ V) = \begin{pmatrix} \lambda & u^H A V \\ 0_{n-1} & V^H A V \end{pmatrix}.$$ 

If $A$ is a Hermitian matrix, then

$$(u \ V)^H A (u \ V) = \begin{pmatrix} \lambda & 0_{n-1} \\ 0_{n-1} & V^H A V \end{pmatrix}.$$
**Theorem**

(Schur’s Triangularization Theorem) Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. There exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{C}^{n \times n}$ such that $A = QTQ^H$ and the diagonal elements of $T$ are the eigenvalues of $A$. Moreover, each eigenvalue $\lambda$ occurs in the sequence of diagonal values a number of $\text{algm}(A, \lambda)$ times.
Proof

The argument is by induction on $n \geq 1$. The base case, $n = 1$, is trivial. So, suppose that the statement is true for matrices in $\mathbb{C}^{(n-1) \times (n-1)}$. Let $\lambda_1 \in \mathbb{C}$ be an eigenvalue of $A$, and let $u$ be an eigenvector that corresponds to this eigenvalue. We have

$$Q^H A Q = \begin{pmatrix} \lambda_1 & u^H A V \\ 0_{n-1} & V^H A V \end{pmatrix},$$

where $Q = (u|V)$ is an unitary matrix.

By the inductive hypothesis, since $V^H A V \in \mathbb{C}^{(n-1) \times (n-1)}$, there exists a unitary matrix $S \in \mathbb{C}^{(n-1) \times (n-1)}$ such that $V^H A V = S^H W S$, where $W$ is an upper-triangular matrix.
Proof (cont’d)

Then, we have

\[ Q^H A Q = \begin{pmatrix} \lambda_1 & u^H V S^H W S \\ 0_{n-1} & S^H W S \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0_{n-1} & W \end{pmatrix}, \]

which shows that an upper triangular matrix \( T \) that is unitarily similar to \( A \) can be defined as

\[ T = \begin{pmatrix} \lambda_1 & 0 \\ 0_{n-1} & W \end{pmatrix}. \]
Proof (cont’d)

Since $T \sim_u A$, it follows that the two matrices have the same characteristic polynomials and therefore, the same spectra and algebraic multiplicities for each eigenvalue.
Example

Let $A \in \mathbb{R}^{3 \times 3}$ be the symmetric matrix

$$A = \begin{pmatrix}
14 & -10 & -2 \\
-10 & -5 & 5 \\
-2 & 5 & 11
\end{pmatrix}$$

whose characteristic polynomial is:

$$p_A(\lambda) = \lambda^3 - 20\lambda^2 - 100\lambda + 2000.$$ 

The eigenvalues of $A$ are $\lambda_1 = 20$, $\lambda_2 = 10$ and $\lambda_3 = -10$. It is easy to see that

$$\mathbf{v}_1 = \begin{pmatrix}
-2 \\
1 \\
1
\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}
-2 \\
-5 \\
1
\end{pmatrix}$$

are eigenvectors that correspond to the eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$, respectively.
Example

The corresponding unit vectors are

\[
\begin{align*}
\mathbf{u}_1 &= \begin{pmatrix}
-\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}}
\end{pmatrix},
\mathbf{u}_2 &= \begin{pmatrix}
\frac{1}{\sqrt{5}} \\
0 \\
\frac{2}{\sqrt{5}}
\end{pmatrix},
\mathbf{u}_3 &= \begin{pmatrix}
\frac{2}{\sqrt{30}} \\
\frac{5}{\sqrt{30}} \\
\frac{1}{\sqrt{30}}
\end{pmatrix}.
\end{align*}
\]

For \( Q = (\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3) \) we have

\[
Q' A Q = Q'(20\mathbf{u}_1 \mathbf{10u}_2 - 10\mathbf{u}_3) = \text{diag}(20, 10, -10).
\]
The Schur decomposition of a square matrix can be computed in \texttt{R} using the function \texttt{Schur} of the package \texttt{Matrix}. For the matrix \( A \) considered before, we can write:

\begin{verbatim}
> A <- matrix(c(14,-10,-2,-10,-5,5,-2,5,11),3,3)
> A
   [,1] [,2] [,3]
[1,]  14 -10  -2
[2,] -10  -5   5
[3,]  -2   5  11
\end{verbatim}
The call to the function Schur

\[ \text{Schur}(A, \text{vectors}=\text{TRUE}) \]

returns a result that has the following components:

$Q$

\[
\begin{bmatrix}
[1,] & [2,] & [3,] \\
1, & 0.3651484 & 0.8164966 & 4.472136 \times 10^{-1} \\
2, & 0.9128709 & -0.4082483 & -3.750263 \times 10^{-19} \\
3, & -0.1825742 & -0.4082483 & 8.944272 \times 10^{-1}
\end{bmatrix}
\]

$T$

\[
\begin{bmatrix}
[1,] & [2,] & [3,] \\
1, & -10 & -1.831868 \times 10^{-15} & -1.160892 \times 10^{-15} \\
2, & 0 & 2.000000 \times 10^{1} & 7.604338 \times 10^{-16} \\
3, & 0 & 0.000000 \times 00 & 1.000000 \times 01
\end{bmatrix}
\]

$EValues$

\[
\begin{bmatrix}
[1] -10 & 20 & 10
\end{bmatrix}
\]
If the vectors parameter is set to FALSE the result includes $T$ and $EValues$. 
Corollary

Let $A \in \mathbb{C}^{n \times n}$ and let $f$ be a polynomial. If $\text{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$ (including multiplicities), then $\text{spec}(f(A)) = \{f(\lambda_1), \ldots, f(\lambda_n)\}$. 
Proof

By Schur’s Triangularization Theorem there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{C}^{n \times n}$ such that $A = U^H TU$ and the diagonal elements of $T$ are the eigenvalues of $A$, $\lambda_1, \ldots, \lambda_n$. Therefore $Uf(A)U^{-1} = f(T)$, and the diagonal elements of $f(T)$ are $f(\lambda_1), \ldots, f(\lambda_m)$. Since $f(A) \sim f(T)$, we obtain the desired conclusion because two similar matrices have the same eigenvalues with the same algebraic multiplicities.
The next statement presents a property of real matrices that admit real Schur factorizations.

**Theorem**

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix. If there exists a orthogonal matrix $U \in \mathbb{R}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n \times n}$ such that $A = U^{-1}TU$, that is, a real Schur factorization, then the eigenvalues of $A$ are real numbers.

**Proof.**

If the above factorization exists we have $T = UAU^{-1}$. Thus, the eigenvalues of $A$ are the diagonal components of $T$ and, therefore, they are real numbers.