Homework 2

Posted: February 20, 2019
Due: March 6, 2019

1. Let \( B = \{x_1, \ldots, x_n\} \) be a finite subset of a metric space \((S, d)\). Prove that
\[
(n - 1) \sum_{i=1}^{n} d(x, x_i) \geq \sum \{d(x_i, x_j) \mid 1 \leq i < j \leq n\}
\]
for every \( x \in S \).

Explain why this inequality can be seen as a generalization of the triangular inequality.

2. Let \((S, d)\) be a metric space and let \( u \in S \) be a fixed element of \( S \).

Define the function \( d_u : S^2 \to \mathbb{R}_{\geq 0} \) by
\[
d_u(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
d(x, u) + d(u, y) & \text{otherwise},
\end{cases}
\]
for \( x, y \in S \). Prove that \( d_u \) is a metric on \( S \).

3. Let \((S, d)\) be a metric space. Prove that \( \sqrt{d} \) and \( \frac{d}{1+d} \) are also metrics on \( S \). What can be said about \( d^2 \)?

4. Let \((S, d)\) be an ultrametric space. Prove that if \( a \geq 0 \), then the mapping \( d_a : S \times S \to \mathbb{R}_{\geq 0} \) defined by \( d_a(x, y) = (d(x, y))^a \) is also an ultrametric metric on \( S \).

5. Let \((S, d)\) be a metric space. Prove that \( d \) is an ultrametric on \( S \) if and only if for every \( a > 0 \) the mapping \( d_a : S \times S \to \mathbb{R}_{\geq 0} \) defined by \( d_a(x, y) = (d(x, y))^a \) for \( x, y \in S \) is a metric on \( S \).

6. Let \((S, d)\) be a dissimilarity space. Prove that \( d \) is an quasi-ultrametric if and only if for every \( u, v \in S \) we have \( B[u, d(u, v)] = B[v, d(u, v)] \).