Convexity and Entropy

Lemma: The function $f : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}$ defined by

$$f_\beta(x) = \frac{1 - x^{1-\beta}}{1 - 2^{1-\beta}}$$

is convex for every $\beta \in [0, 1) \cup (1, \infty)$.

Proof: We have

$$\begin{aligned} f'_\beta(x) &= \frac{-(1-\beta)x^{-\beta}}{1 - 2^{1-\beta}}, \\ f''_\beta(x) &= \frac{\beta(1-\beta)x^{-\beta-1}}{1 - 2^{1-\beta}}. \end{aligned}$$

Note that $1 - 2^{1-\beta} > 0$ if $\beta > 1$ and $1 - 2^{1-\beta} < 0$ if $\beta < 1$; in each case $f_\beta(x) < 0$, so the function f_β is concave for every $\beta \in (0, 1) \cup (1, \infty)$.

Theorem: Let $H_\beta : S_n \longrightarrow \mathbb{R}$ be defined as

$$H_\beta(\mathbf{p}) = \frac{1}{1 - 2^{1-\beta}} \left(1 - \sum_{i=1}^{n} p_i^\beta \right),$$

for $\mathbf{p} \in S_n$, where S_n is the probability simplex in $\mathbb{R}^n$. The following statements hold:

1. $H_\beta(\mathbf{p}) = \sum_{i=1}^n f_\beta(p_i)$, where f_β is the function introduced in the previous lemma;

2. for every $\mathbf{p} \in S_n$ we have $H_\beta(\mathbf{p}) \leqslant \frac{1-n^{1-\beta}}{1-2^{1-\beta}}$ and the maximum value of $H_\beta(\mathbf{p})$ is obtained when $p_1 = \cdots = p_n = \frac{1}{n}$;

3. we have $\lim_{\beta \to 1} H_\beta(\mathbf{p}) = H(\mathbf{p})$, where $H(\mathbf{p})$ is the Shannon entropy of $\mathbf{p}$.

Proof: For $\beta = 2$, we have

$$H_2(\mathbf{p}) = 2(1- \parallel p \parallel^2).$$

This quantity is known as the *Gini index*.

The concavity of f_β implies

$$f_\beta\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant \sum_{i=1}^{n} p_i f_\beta(x_i)$$

for $\mathbf{p} \in S_n$. Taking $x_i = \frac{!}{p_i}$ yields

$$\begin{aligned} f_\beta(n) &\geqslant \sum_{i=1}^{n} p_i \frac{1 - \frac{1}{p_i^{1-\beta}}}{1 - 2^{1-\beta}} = \frac{1}{1 - 2^{1-\beta}} \sum_{i=1}^{n} (p_i - p_i^\beta) \\ &= \frac{1}{1 - 2^{1-\beta}} \left(1 - \sum_{i=1}^{n} p_i^\beta\right) = H_\beta(\mathbf{p}). \end{aligned}$$

Therefore $H_\beta(\mathbf{p}) \leqslant \frac{1-n^{1-\beta}}{1-2^{1-\beta}}$. The argument for the remaining parts are left as exercises.