Convex Sets and Functions (part III)

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1 The epigraph and hypograph of functions

2 Constructing Convex Functions

Definition

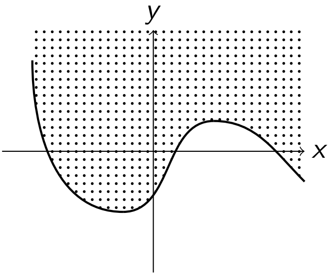
Let $(S, \mathcal{O})$ be a topological space and let $f : S \longrightarrow \hat{\mathbb{R}}$ be a function. Its epigraph is the set

$$\mathrm{epi}(f) = \{(x, y) \in S \times \mathbb{R} \mid f(x) \leqslant y\}.$$

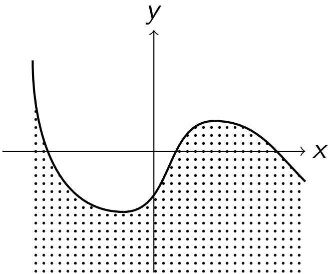
The hypograph of f is the set

$$\mathrm{hyp}(f) = \{(x, y) \in S \times \mathbb{R} \mid y \leqslant f(\mathbf{x})\}.$$

The epigraph of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the dotted area in $\mathbb{R}^2$ located above the graph of the function f; the hypograph of f is the dotted area below the graph.



(a)



(b)

Note that the intersection

$$\text{epi}(f) \cap \text{hyp}(f) = \{(x, y) \in S \times \mathbb{R} \mid y = f(x)\}$$

is the graph of the function f.
If $f(x) = \infty$, then $(x, \infty) \notin \text{epi}(f)$. Thus, for the function f_∞ defined by $f_\infty(x) = \infty$ for $x \in S$ we have $\text{epi}(f_\infty) = \emptyset$.

Theorem

Let $f : S \longrightarrow \mathbb{R}$ be a function defined on the convex subset S of a real linear space L. Then, f is convex on S if and only if its epigraph is a convex subset of $S \times \mathbb{R}$; f is concave if and only if its hypograph is a convex subset of $S \times \mathbb{R}$.

Proof

Let f be a convex function on S. We have $f((1-t)\mathbf{x}+t\mathbf{y}) \leqslant (1-t)f(\mathbf{x})+tf(\mathbf{y})$ for every $\mathbf{x},\mathbf{y} \in S$ and $t \in [0,1]$. If $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2) \in \text{epi}(f)$ we have $f(\mathbf{x}_1) \leqslant y_1$ and $f(\mathbf{x}_2) \leqslant y_2$. Therefore,

$$\begin{aligned} f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) &\leqslant (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2) \\ &\leqslant (1-t)y_1 + ty_2, \end{aligned}$$

so $((1-t)\mathbf{x}_1 + t\mathbf{x}_2, (1-t)y_1 + ty_2) = (1-t)(\mathbf{x}_1, y_1) + t(\mathbf{x}_2, y_2) \in \text{epi}(f)$ for $t \in [0,1]$. This shows that $\text{epi}(f)$ is convex.

Proof (cont'd)

Conversely, suppose that epi(f) is convex, that is, if $(\mathbf{x}_1, y_1) \in \text{epi}(f)$ and $(\mathbf{x}_2, y_2) \in \text{epi}(f)$, then

$$(1-t)(\mathbf{x}_1, y_1) + t(\mathbf{x}_2, y_2) = ((1-t)\mathbf{x}_1 + t\mathbf{x}_2, (1-t)y_1 + ty_2) \in \text{epi}(f)$$

for $t \in [0,1]$. By the definition of the epigraph, this is equivalent to $f(\mathbf{x}_1) \leqslant y_1$, $f(\mathbf{x}_2) \leqslant y_2$ implies $f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) \leqslant (1-t)y_1 + ty_2$. Choosing $y_1 = f(\mathbf{x}_1)$ and $y_2 = f(\mathbf{x}_2)$ yields $f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) \leqslant (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2)$, which means that f is convex.
The second part of the theorem follows by applying the first part to the function $-f$.

If $f_1, \ldots, f_k$ are k convex functions on a linear space that any positive combination $a_1 f_1 + \cdots + a_k f_k$ is a convex function.

Theorem

If f, g are convex functions defined on a real linear space L, then the function h defined by $h(x) = \max\{f(x), g(x)\}$ for $x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ is a convex function.

Proof

Let $t \in [0,1]$ and let $x_1, x_2 \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$.
We have

$$\begin{aligned}
h((1-t)x_1 + tx_2) &= \max\{f((1-t)x_1 + tx_2), g((1-t)x_1 + tx_2)\} \\
&\leqslant \max\{(1-t)f(x_1) + tf(x_2), (1-t)g(x_1) + tg(x_2)\} \\
&\leqslant (1-t)\max\{f(x_1), g(x_1)\} + t\max\{f(x_2), g(x_2)\} \\
&= (1-t)h(x_1) + th(x_2),
\end{aligned}$$

which implies that h is convex.

Theorem

Let C be a convex subset of $\mathbb{R}^n$, b be a number in $\mathbb{R}$, and let $\mathcal{F} = \{f_i \mid f_i : C \longrightarrow \mathbb{R}, i \in I\}$ be a family of convex functions such that $f_i(\mathbf{x}) \leqslant b$ for every $i \in I$ and $\mathbf{x} \in C$. Then, the function $f : C \longrightarrow \mathbb{R}$ defined by

$$f(\mathbf{x}) = \sup\{f_i(\mathbf{x}) \mid i \in I\}$$

for $\mathbf{x} \in C$ is a convex function.

Proof

Since the family of function $\mathcal{F}$ is upper bounded, the definition of f is correct. Let $\mathbf{x}, \mathbf{y} \in C$. We have $(1-t)\mathbf{x} + t\mathbf{y} \in C$ because C is convex. For every $i \in I$ we have $f_i((1-t)\mathbf{x} + t\mathbf{y}) \leqslant (1-t)f_i(\mathbf{x}) + tf_i(\mathbf{y})$. The definition of f implies $f_i(\mathbf{x}) \leqslant f(\mathbf{x})$ and $f_i(\mathbf{y}) \leqslant f(\mathbf{y})$, so $(1-t)f_i(\mathbf{x}) + tf_i(\mathbf{y}) \leqslant (1-t)f(\mathbf{x}) + tf(\mathbf{y})$ for $i \in I$ and $t \in [0,1]$. The definition of f implies $f((1-t)\mathbf{x} + t\mathbf{y}) \leqslant (1-t)f(\mathbf{x}) + tf(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in C$ and $t \in [0,1]$, so f is convex on C.

Definition

Let $f : S \longrightarrow \mathbb{R}$ be a convex function and let $g_{\mathbf{x}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $g_{\mathbf{x}}(\mathbf{y}) = \mathbf{y}'\mathbf{x} - f(\mathbf{x})$.
The conjugate function of f is the function $f^* : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f^*(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbb{R}^n} g_{\mathbf{x}}(\mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^n$.

Note that for each $\mathbf{x} \in \mathbb{R}^n$ the function $g_{\mathbf{x}} = \mathbf{y}'\mathbf{x} - f(\mathbf{x})$ is a convex function of $\mathbf{y}$. Therefore, f^* is a convex function.

Example

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = e^x$. We have $g_x(y) = yx - e^x$. Note that:

- if $y < 0$, each such function is unbounded, so $f^*(y) = \infty$;
- if $y = 0$, $f^*(0) = \sup_x e^{-x} = 0$;
- if $y > 0$, g_x reaches its maximum when $x = \ln y$, so $f^*(y) = y \ln y - y$.

Thus, $\text{Dom}(f^*) = \mathbb{R}_{\geqslant 0}$ and $f^*(y) = y \ln y - y$ (with the convention $0\infty = 0$.

Example

Let a be a positive number and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = \frac{a}{2}x^2$. We have $g_x(y) = yx - \frac{a}{2}x^2$ and

$$\sup_{x \in \mathbb{R}} \left(yx - \frac{a}{2}x^2 \right) = \frac{1}{2a}y^2$$

and therefore

$$f^*(y) = \frac{1}{2a}y^2.$$

Example

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be $f(\mathbf{x}) = \parallel \mathbf{x} \parallel$, where $\mathbf{x}$ is the Euclidean norm on $\mathbb{R}^n$.
We have $g_{\mathbf{x}} = \mathbf{y}'\mathbf{x} - \parallel \mathbf{x} \parallel$.
If $\parallel \mathbf{y} \parallel \leqslant 1$, taking into account that $y'\mathbf{x} \leqslant \parallel \mathbf{x} \parallel \parallel \mathbf{y} \parallel$, it follows that $\mathbf{y}'\mathbf{x} \leqslant \parallel \mathbf{y} \parallel$, so $y'\mathbf{x} - \parallel \mathbf{x} \parallel \leqslant 0$. Therefore, $\mathbf{x} = \mathbf{0}_n$ maximizes $\mathbf{y}'\mathbf{x} - \parallel \mathbf{x} \parallel$, so $f^*(\mathbf{y}) = 0$.
If $\parallel \mathbf{y} \parallel > 1$, there is a $\mathbf{z}$ such that $\parallel \mathbf{z} \parallel \leqslant 1$ and $\mathbf{y}'\mathbf{z} > 1$. It suffices to choose $\mathbf{z}$ such that

$$\frac{1}{\parallel \mathbf{y} \parallel} < \parallel \mathbf{z} \parallel \leqslant 1.$$

Choosing $\mathbf{x} = t\mathbf{z}$ and letting $t \to \infty$ we have $\mathbf{y}'\mathbf{x} - \parallel \mathbf{x} \parallel = t(\mathbf{y}'\mathbf{z} - \parallel \mathbf{z} \parallel) \to \infty$. Thus, we have

$$f^*(\mathbf{y}) = \begin{cases} 0 & \text{if } \parallel \mathbf{y} \parallel \leqslant 1, \\ \infty & \text{otherwise.} \end{cases}$$