Homework 2
posted March 8, 2021
due March 24, 2021

Let \( B_n = \{0, 1\}^n \) and let \( K \subseteq B_n \). The sequence of Chow parameters of \( K \) is \( \text{chow}(K) = (c_1, \ldots, c_n, c_K) \in n^n \) defined as \( c_K = |K| \) and \( c_i = |\{x \in K \mid x_i = 1\}| \). For example, for \( n = 4 \) and \( K = \{(0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\} \) we have \( \text{chow}(K) = (0, 1, 2, 3, 3) \).

Two subsets \( K, G \) of \( B_n \) are equipollent if they have the same Chow parameters.

The subsets \( K \) and \( B_n - K \) are linearly separable if there exists a pair \( (w, t) \in \mathbb{R}^n \times \mathbb{R} \) such that
\[
K = \{x \in B_n \mid w'x \geq t\} \quad \text{and} \quad B_n - K = \{x \in B_n \mid w'x < t\}.
\]

We say that \( K \) is linearly separable if \( K \) and \( B_n - K \) are linearly separable.

1. Let \( K \subseteq B_n \). Prove that \( \text{chow}(K) = (\sum_{x \in K} x, |K|) \).

2. A diagonal of \( B_n \) is a pair \( (u, v) \in B_2^n \) such that \( u = 1_n - v \), where \( 1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \). Prove that if \( K \) is a linearly separable subset of \( B_n \) that contains a diagonal of \( B_n \), then it contains a point of every other diagonal of \( B_n \).

3. The optimization problem of the separable data case that seeks to determine a separating hyperplane in \( \mathbb{R}^n \) can be transformed into an equivalent optimization problem in \( \mathbb{R}^{n+1} \) that seeks to identify a separating subspace. Given a data set \( s = ((x_1, y_1), \ldots, (x_m, y_m)) \) prove that there exists \( \mathbf{r} \in \mathbb{R}^n \) such that \( s \) is separable by a hyperplane if and only if the set \( \tilde{s} = ((x_1 + \mathbf{r}, y_1), \ldots, (x_m + \mathbf{r}, y_m)) \) is separable be a subspace \( M \) of \( \mathbb{R}^n \).

4. Consider the data set \( D \) in \( \mathbb{R}^2 \) shown in Figure 1, where \( C \) is a circle centered in \((6, 4)\) having radius 3. Define a transformation \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( \phi(D) \) is linearly separable.

5. There are 16 functions of the form \( f : \{0, 1\}^2 \rightarrow \{0, 1\} \). For each such function consider the sequence \( S_f = ((x_1, y_1), \ldots, (x_4, y_4)) \), where
Figure 1: Non-linearly separable data; positive examples are filled circles.

\[ x_i \in \{0, 1\}^2 \text{ and } \]
\[ y_i = \begin{cases} -1 & \text{if } f(x_i) = 0, \\ 1 & \text{if } f(x_i) = 1 \end{cases} \]

for \( 1 \leq i \leq 4 \). For how many of these functions is \( S_f \) linearly separable?