Homework 4

Posted: November 9, 2020
Due: November 23, 2020

1. Define the predicate \( P_k(x) \) as
\[
P_k(x) = \begin{cases} 
1 & \text{if } \Phi_x(x) = k, \\
0 & \text{otherwise.}
\end{cases}
\]
Prove that \( P_k \) is not computable.

2. Let \( A, B \) be two subsets of \( \mathbb{N} \). Define the sets \( A \oplus B \) and \( A \otimes B \) as
\[
A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\} \\
A \otimes B = \{(x, y) \mid x \in A \text{ and } y \in B\}.
\]
Prove that:
(a) \( A \oplus B \) is recursive if and only if \( A \) and \( B \) are both recursive;
(b) if \( A \) and \( B \) are non-empty, then \( A \otimes B \) is recursive if and only if \( A \) and \( B \) are both recursive.

3. Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be a unary function. Prove that \( f \) is computable if and only if the set \( \{2^x \cdot 3^f(x) \mid x \in \text{Dom}(f)\} \) is recursively enumerable.

4. If \( A \leq_m B \), prove that \( \overline{A} \leq_m \overline{B} \). Here \( \overline{C} \) is the complement of \( C \), that is, \( \overline{C} = \mathbb{N} - C \).

5. Prove that the set \( A = \{x \mid \text{Dom}(\Phi_x) \neq \emptyset\} \) is recursively enumerable, but not recursive.