1. Define the predicate $P_k(x)$ as

$$P_k(x) = \begin{cases} 1 & \text{if } \Phi_x(x) = k, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $P_k$ is not computable.

2. Let $A, B$ be two subsets of $\mathbb{N}$. Define the sets $A \oplus B$ and $A \otimes B$ as

$$A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$$
$$A \otimes B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Prove that:

(a) $A \oplus B$ is recursive if and only if $A$ and $B$ are both recursive;

(b) if $A$ and $B$ are non-empty, then $A \otimes B$ is recursive if and only if $A$ and $B$ are both recursive.

3. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a unary function. Prove that $f$ is computable if and only if the set $\{2^x \cdot 3^{f(x)} \mid x \in \text{Dom}(f)\}$ is recursively enumerable.

4. If $A \leq_m B$, prove that $\overline{A} \leq_m \overline{B}$. Here $\overline{C}$ is the complement of $C$, that is, $\overline{C} = \mathbb{N} - C$.

5. Prove that the set $A = \{x \mid \text{Dom}(\Phi_x) \neq \emptyset\}$ is recursively enumerable, but not recursive.