THE SPINOR SPANNER

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1. Introduction. Consider a wrench, which is an object asymmetrical enough so that the result of any proper rotation performed on it is easily recognized. Rotate the wrench through a full 360° turn about an axis. Has it returned to its original state? Physical and geometric intuition both say "yes", yet the calculus of spinors, which models the quantum mechanical behavior of neutrons, predicts that the answer would be "no" if the wrench were a neutron, or any other Fermion, a particle with half integral spin. More striking still, the predicted answer is "yes" for two full turns (720°) about the same axis. No experiment has yet been performed to verify these predictions, because beam splitters and interferometers for beams of polarized neutrons do not yet exist, but several such experiments have been imagined [**1**], [**2**]. There is, however, an easy experiment with an analogous outcome. P. A. M. Dirac invented it to lessen, in lectures, the implausibility of the neutron's predicted behavior [**3**]. Consider the wrench again, which Dirac would have called by its English name, a spanner, hence a spinor spanner because of the use to which he put it. Attach it by three cords to the walls of the room. (See the solid lines in Fig. 1.)

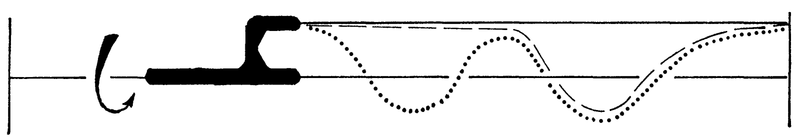

FIG. 1

When we turn the wrench through 360° the cords become tangled (the dashed lines in Fig. 1); no tampering can undo that tangle as long as the wrench is fixed. After two full turns (the dotted line in Fig. 1) the snarl seems worse but is not. Before reading further, find a wrench, perform the experiment, and convince yourself of the striking fact that after two full turns the cords are essentially untangled. The geometry of the spinor spanner is the key to Piet Hein's topological game Tangloids, described by Martin Gardner in the *Scientific American* [**9**], and to an ingenious device invented and patented by D. A. Adams which allows a rotating platform to be connected to a stationary base with a flexible cable without using slip rings or rotary joints [**8**].

I first saw the spinor spanner demonstrated by Norman Ramsey, a physicist, while I was a graduate student. In this paper I shall explain in mathematical terms why the spinor spanner works, and indicate how that explanation can be couched

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in language suitable for mathematics clubs and more general mathematically naive audiences. Someday, I should like to make a movie of the spinor spanner.

We are about to show that the fundamental group G of $SO(3)$, the group of proper rotations of Euclidean 3-space, is of order 2, and to exploit the proof to find a method for untangling the cords. Since Fermions correspond to representations of the double covering group of $SO(3)$ which do not factor through $SO(3)$ itself, the fact that the order of G is 2 really accounts both for the spinor spanner and for the neutron's behavior.

2. Homotopy. Let X be a topological space and x_0 a fixed point in X. A naive audience could think of X as a smooth part of some Euclidean space, say the surface of a sphere, or a solid torus, or an annulus. A *loop* in X is a continuous function P: $[0,1] \to X$ for which $P(0) = P(1) = x_0$. If you think of X as a park then a loop may be thought of as the record of an hour's walk in X, starting and ending at x_0. Be sure to distinguish this precise usage from the more customary meaning of "closed path in a park." The latter is the image of the function P. Two loops P and Q are **homotopic,** written $P \sim Q$, when one can be continuously deformed into the other. Formally, $P \sim Q$ when there is a continuous f: $[0,1] \times [0,1] \to X$ for which $f(0,s) = f(1,s) = x_0$, $f(t,0) = P(t)$ and $f(t,1) = Q(t)$. Informally, suppose that you walk your dog in X: you follow P while he follows Q. Then $P \sim Q$ means that when the walk is over the leash joining the two of you can be pulled in without encountering any parklike obstacles, trees or lakes, which you and your dog passed on opposite sides of. This interpretation makes clear the importance of the direction in which you traverse the curve which is the image of P. If P is a sense preserving reparametrization of Q then $P \sim Q$. *The loop corresponding to the lazy man's walk* is the constant loop 0 defined by $0(t) = x_0$ for all t.

Now let us consider taking two walks in succession. We shall denote "P followed by Q" by "$P \oplus Q$". As a function, $P \oplus Q$ is defined by

$$P \oplus Q(t) = \begin{cases} P(2t) & \text{if} \quad 0 < t \leqq 1/2 \\ Q(2t-1) & \text{if} \quad 1/2 \leqq t \leqq 1. \end{cases}$$

It is intuitively clear and not hard to prove that homotopy is an equivalence relation, that the homotopy class of $P \oplus Q$ depends only on the classes of P and of Q, and that the set of homotopy classes is a group under $\oplus$. Details can be found in many topology texts (for example, [5] and [7]). The group is not usually abelian, but I have found additive notation less confusing than multiplicative for naive audiences. Observe that 0 is the group identity: $P \oplus 0 \sim P$. Our job now is to find the inverse of P, the solution to $P \oplus ? \sim 0$. The dog walking analogy can lead us to a good guess. If you are lazy while your dog follows P then his leash will be tangled when he returns, unless, by chance, $P \sim 0$. How could you untangle the leash? If the dog is intelligent the answer is easy: ask him to retrace his steps. That is, if we define the loop $-P$ by $(-P)(t) = P(1-t)$ then $P \oplus (-P) \sim 0$.

3. Pasting. The topological spaces we can visualize as smooth parts of 2- or 3-space are too simple to help us analyze the spinor spanner. We need a method for studying homotopy in more complicated ones.

If we take a square piece of paper and paste together a pair of parallel sides, top to top and bottom to bottom, we have made a cylinder. We can study homotopy on the cylinder without actually pasting the square, as long as we remember that points along one edge are identified with corresponding points on the other. The idea of "pasting" can be made precise using quotient topologies, but we have no need for that much sophistication. For naive audiences it is instructive to mention the various spaces which can be obtained by pasting edges of a rectangle. They are the cylinder, the Möbius strip, the torus, the Klein Bottle, and, finally, the projective plane. The identifications which lead to these are symbolically indicated in Figs. 2.1–2.5 respectively, in which some loops are sketched as well. The Klein Bottle and the projective plane cannot actually be constructed in 3-space but we can study them nevertheless. Fig. 2.5 suggests a more symmetrical view of the projective plane Π. Since each pair of opposite points of the square is pasted, the corners assume no special role. We can build Π from a disk Δ by pasting together each pair of antipodal points on the rim: in Fig. 3, these are pairs (A, A'), (B, B'), (C, C'), etc.

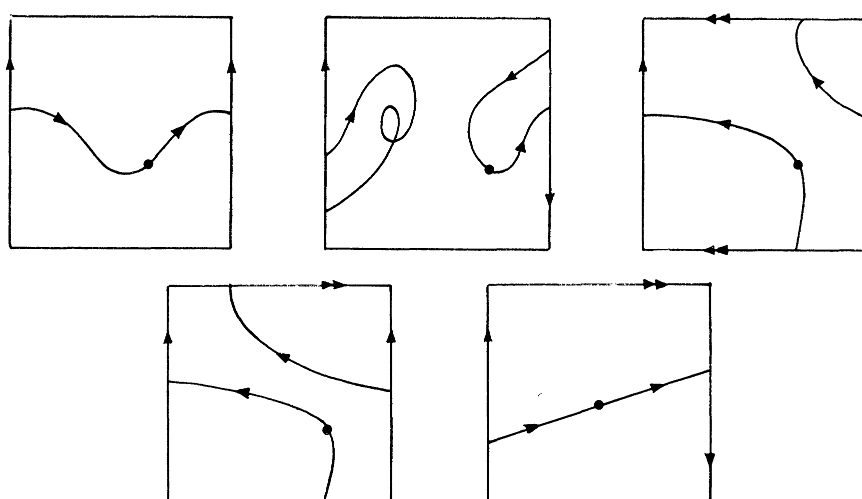

FIG. 2 (2.1–2.5)

Let x_0 be the center and L a directed diameter of Δ. Since the ends of L are identified when we build Π we can consider the loop P in Π which begins at x_0, follows L to the rim of Δ and then returns to x_0 along the other half of L. We show next that $P \nsim 0$; to do so we use a homeomorphic copy or model of Π. Start with the disk Δ and stretch it to form a closed hemisphere. Now consider the spherical surface Σ of which Δ is a part. If we paste together each pair of antipodal points of Σ, then Π will result. To see this, paste first all the antipodal pairs one member or which lies in the interior of Δ. That yields the hemisphere into which Δ was stretched. The rest of the pasting, of the pairs on the equator, is just what to do to Δ to build Π. In this model for Π the north and south poles n and s of Σ paste together to make x_0. In Σ there is a unique continuous curve S which starts at n and which becomes P when Σ is pasted to form Π, namely, the appropriate meridian joining n to s. That curve is not a loop in Σ. If P were homotopic to 0 in Π we could lift that homotopy to Σ and so construct a continuous deformation in Σ of S to the constant loop at n during which the endpoints n and s of S remained fixed. Since such a deformation is clearly impossible, $P \nsim 0$ in Π. We can see too that $P \oplus P \sim 0$, because $P \oplus P$ is the result in Π of pasting a great circle through n and s in Σ. That great circle easily shrinks to the constant loop at n in Σ. But to untangle cords later, we must now show in another way that $P \oplus P \sim 0$. Consider again our first model for Π, obtained by pasting pairs of opposite points on the rim of Δ.

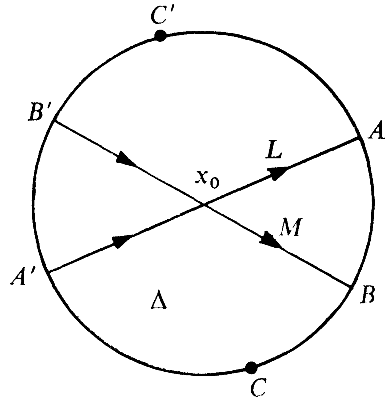

FIG. 3

Let M be another directed diameter of Δ, and let Q follow M in Π as P follows L (see Fig. 3). We can rotate L in Δ until it coincides with M; this rotation is a continuous deformation in Π of P to Q. If we take for M the diameter L with its direction reversed then Q is $-P$, so $P \oplus P \sim P \oplus (-P) \sim 0$. The projective plane thus surrounds a peculiar kind of hole. If you travel around it twice in the same direction you've not gone around it at all. That is analogous to what happens to the spinor spanner. In each case doubling something makes it vanish. But with the techniques of homotopy and pasting, we can do better than produce an analogy for the spinor spanner. We can predict and explain its behavior.

4. The topology of $SO(3)$. Let Ω be the space of all possible configurations of the wrench. A point $\omega \in \Omega$ is thus the result of a particular proper rotation. Remember, it is the configuration we are talking about, not the means by which the wrench came to that configuration. It is intuitively clear that Ω is a nice topological space. Our complete turn of the wrench about an axis corresponds to a loop P in Ω which begins and ends at the initial configuration ω_0. We shall show $P \nsim 0$ but $P \oplus P \sim 0$ in Ω and then show how the homotopy which shrinks $P \oplus P$ to 0 tells us how to untangle cords.

We begin by building a model of Ω. Replace the wrench by the surface of a sphere Σ centered at the origin. Then each $\omega \in \Omega$ can be identified with a map from Σ to itself defined by letting $\omega(\sigma) =$ the position of $\sigma \in \Sigma$ when Σ is moved to configuration ω. As a map, ω preserves distances and the orientation of spherical triangles. We next show, in two ways, that every such map has a fixed point. Since ω extends to a proper linear isometry of $\mathbb{R}^3$ the roots of its cubic characteristic polynomial have product 1 and each is of absolute value 1. Thus those roots are 1, $e^{i\theta}$, $e^{-i\theta}$ for some θ. Since 1 is a root, 1 is an eigenvalue and ω has a fixed point. This argument clearly works in $\mathbb{R}^n$ if and only if n is odd.

Here is a second proof in $\mathbb{R}^3$, suitable for audiences who know no linear algebra. Let Σ have circumference 2. For $x, y \in \Sigma$ let $\mu(x, y)$ be the least great circle distance between x and y. The function $f\colon \Sigma \to \mathbb{R}$ defined by $f(x) = \mu(x, \omega x)$ is continuous and so assumes its minimum value $\delta \geqq 0$ at some $a \in \Sigma$. If $\delta = 0$ then $\omega a = a$ and ω has the fixed point we desire. We shall show next that $\delta > 0$ implies $\delta = 0$. Suppose $\delta > 0$. Since ω is proper it cannot map every point to its antipode. Thus $\delta < 1$, so we can find a hemisphere H containing both a and ωa. In H draw the great circle Γ joining a to ωa; it has length δ. Now draw two circles C and D centered at a and ωa respectively; make them so small that they lie in H and do not overlap. Let c be the intersection of C and Γ and d the intersection of D and the continuation of Γ. Since $\mu(a, c) = \mu(\omega a, \omega c)$, $\omega c \in D$. But every point on D except d is less than δ units from c, so $\omega c = d$. Now let n and s be the poles for which Γ lies on the equator. Then $\mu(a, n) = \mu(c, n) = 1/2$ so $\mu(\omega a, \omega n) = \mu(\omega c, \omega n) = 1/2$. Therefore $\omega n = n$ or s. But $\omega n = s$ is impossible because ω preserves the orientation of the spherical triangle $a\,c\,n$. Thus $\omega n = n$, n is a fixed point, and $\delta = 0$. That is, a must have been fixed to begin with.

Suppose $\omega \neq \omega_0$. Then ω has exactly two fixed points n_ω, s_ω; which lie at opposite ends of a diameter of Σ, and Σ can be brought to configuration ω by a rotation of r radians about the axis n_ω, s_ω. We wish to consider rotations which are counterclockwise when we look down on n_ω from outer space; this is the familiar right hand rule. In lectures I use an inflatable globe to show a counterclockwise rotation of 102° about the axis joining Bermuda to Perth, Australia, moves Duluth to the Panama Canal. Since a clockwise rotation about an axis is a counterclockwise rotation about the same axis with its north and south poles interchanged, and since rotations through r and $r - 2\pi$ radians about an axis lead to the same ω, we can describe an $\omega \neq \omega_0$ by giving a vector $m(\omega) \neq 0$ with length $\| m(\omega) \| \leqq \pi : m(\omega)$

points toward n_ω from the origin, and $\| m(\omega) \| = r$. If we set $m(\omega_0) = 0$, the range of m is the solid ball B of radius π centered at 0. The function which inverts m is one to one except when $\| V \| = \pi$, for rotations through π radians about V and $-V$ lead to the same ω. Thus Ω is modeled by the space X which results *when we paste together each pair of antipodal points on the surface of the solid ball* B, because $m\colon \Omega \to X$ is a homeomorphism; one to one, onto, continuous, and with a continuous inverse. The loop P in X which corresponds to a full turn of the wrench about an axis L starts at the center of B, moves out along L to the surface and returns to the center along the other half of L. It is analogous to the loop with the same name we have just studied in Π. In fact, Π is a subspace of X in a natural way, so that the two loops we have named "P" coincide. Since $P \sim -P$ in Π, $P \sim -P$ in X. For those who like formulas, we give one for that homotopy. Let Δ be the intersection of B with the x, z plane and L the directed diameter which extends to the directed x axis. The homotopy which interests us rotates L in Δ to change P to $-P$. The matrix for a right handed rotation through r radians about the axis in the x, z plane which makes an angle of θ radians with L is

$$f(r,\theta) = \begin{bmatrix} \cos^2\theta + (\cos r)\sin^2\theta & -(\sin r)\sin\theta & (1-\cos r)\sin\theta\cos\theta \\ (\sin r)\sin\theta & \cos r & -(\sin r)\cos\theta \\ (1-\cos r)\sin\theta\cos\theta & (\sin r)\cos\theta & \sin^2\theta + (\cos r)\cos^2\theta \end{bmatrix}.$$

The function f is continuous on $[0, 2\pi] \times [0, \pi]$, $f(\,\cdot\,, 0)$ is the loop P, and $f(\,\cdot\,, \pi)$ is the loop $-P$, so f is our homotopy in Ω.

To prove $P \nsim 0$ in X we cannot merely use the fact that P lives in the subspace Π of X, for although no deformation of P to 0 is possible inside that subspace one might be possible in X. To rule that out we need a new model for Ω analogous to our second model for Π, the one we built by pasting antipodal points on the 2-sphere Σ. Let Φ be the 3-sphere in real 4-space. We can stretch B so that it covers a hemisphere of Φ. Then Ω results when we paste antipodal pairs in Φ, since B results when we paste first those pairs one member of which is interior to B. In this model the north and south poles of Φ paste together to make ω_0. Now the proof that $P \nsim 0$ in Ω proceeds as it did for Π. In technical terms, we have just constructed and then used a simply connected covering space Φ for Ω.

5. Untangling cords. To exploit the fact that $P \sim -P$, and hence that $P \oplus P \sim 0$ in Ω, we must model Ω and loops in it one more way. Consider two concentric spheres; call the inner one the globe (or the wrench, or the neutron) and the outer one the edge of the universe. Suppose the distance between the spheres is 1. Cords, as many of them as we wish to attach, lie initially along radii joining the globe to the edge of the universe. Pack the space between the globe and the edge of the universe with concentric spherical shells Σ_t, where $t \in [0, 1]$ measures the distance of Σ_t from Σ_0. Each cord is attached to Σ_t where they meet.

Imagine that the shells can slide relative to each other. Let R be any loop in Ω starting and ending at ω_0; suppose we manipulate the globe Σ_1 so that at time t it is at $R(t)$. Then the cords cause the intermediate shells to record R: at time t, Σ_t is in configuration $R(t)$ (Σ_t). A homotopy $R \sim Q$ of paths in Ω is a function f: $[0,1] \times [0,1] \to \Omega$ satisfying the conditions listed earlier. If we now manipulate the shells Σ_t so that at time s, shell Σ_t is at position $f(t,s)$ (Σ_t) we shall have deformed the cords, which initially recorded R, to a record of Q. Thus when P is the loop corresponding to a full turn about an axis the homotopy $P \oplus P \sim 0$ really tells us that our cords can be untangled, even if we started with many more than three.

Because $P \nsim 0$, no manipulation of the intermediate shells can untangle the cords after one full turn. It is true and slightly subtler that they cannot be untangled at all [6].

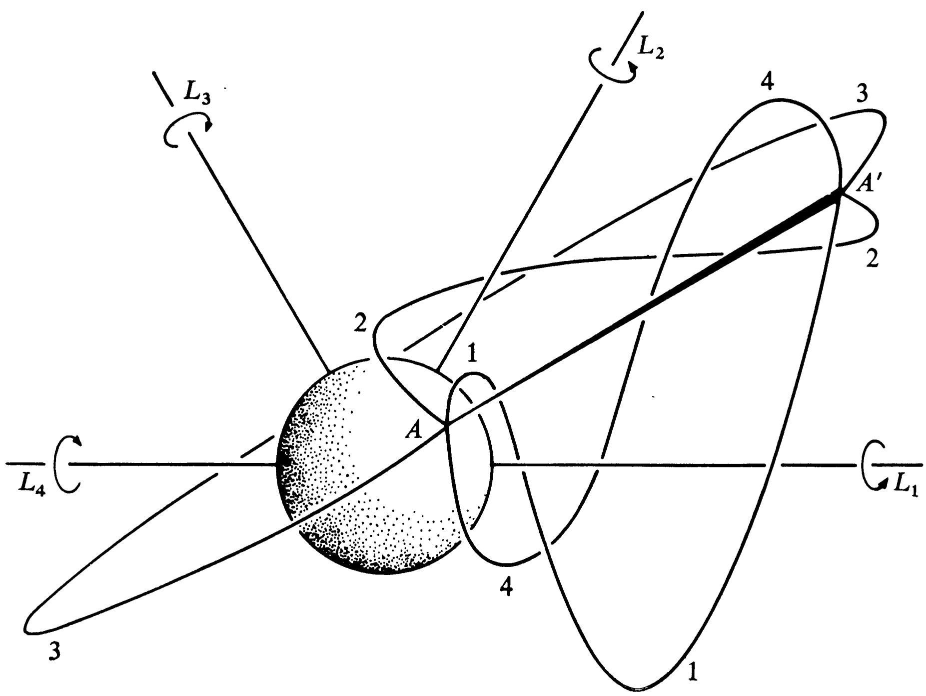

FIG. 4

Let us close by seeing just how the particular homotopy we have studied untangles the cords after two full turns. To convert $P \oplus P$ to 0 we first deform the second summand P to $-P$, or, in other words, deform the result of a full right handed turn about L to the result of a full left handed turn. We do that by rotating L, the axis of the turn, in the subspace Δ of B, so that it reverses its direction. In Fig. 4 we sketch what happens to one of the cords between Σ_1 and $\Sigma_{1/2}$, the one which lies initially along axis AA'. When the globe executes a full turn about L_i the cord assumes position i. As i varies from 1 to 4, L_i rotates counterclockwise through π radians in the plane of the paper. That operation simultaneously loops the pictured cord and all others on the right over and behind the wrench and those on the left under and in front. That is easier to do than to describe: try it. It really untangles cords. With a little practice it makes a good lecture demonstration or conversation piece, a magic trick which is not magic, but which reflects a fundamental yet little known property of the space in which we live. The analogy between the spinor spanner and the neutron suggests that the state of the latter depends not only on its position and momentum but on which of two topologically distinct ways it is tied to its surroundings. A full turn about an axis leaves its position and momentum unchanged but reverses *its topological relation to the rest of the universe.*

Acknowledgments. Some of the ideas in this paper I explored in conversation with M. Artin, F. C. Cunningham, Jr., M. Gaffney, A. M. Gleason, N. Stein, and N. Ramsey. I am indebted to the Bryn Mawr College Chapter of Sigma Xi and to West Chester State College, West Chester, Pa., where I spoke on the spinor spanner. Final thanks go to Ms. Jessica Bolker, who built the large model wrench I turn while I talk.

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