1. Exercise 5.4

No. For example, let $A = \{0^n1^n|n \geq 0\}$ and $B = \{0^n1^m|n, m \geq 0\}$. We will show that $A \leq_m B$ even though $A$ is not regular and $B$ is regular. A mapping reduction $f$ from $A$ to $B$ is computed by the Turing machine $F$ given by

$$F = \text{"On input } w \in \{0,1\}^*, \text{1. Determine if } w \text{ belongs to } A. \text{ (Since } A \text{ is decidable, } F \text{ can do this with no further information.)}$$

$$2. \text{If } w \text{ belongs to } A, \text{ then output } 01. \text{ If } w \text{ does not belong to } A, \text{ then output } 10."$$

(In fact, the same argument shows that if $A$ is any decidable language and $B$ is any language other than $\emptyset$ and $\Sigma^*$, then $A \leq_m B$.)

2. Problem 5.22

A is Turing-recognizable if and only if $A \leq_m A^M_T$.

**Proof:** First suppose that $A$ is Turing-recognizable and let $M$ be a Turing machine that recognizes $A$. The function $f$ defined by $f(w) = \langle M, w \rangle$ is a reduction from $A$ to $A^M_T$ because it is obviously computable and we have

$$w \in A \text{ iff } M \text{ accepts } w \text{ iff } \langle M, w \rangle \in A^M_T \text{ iff } f(w) \in A^M_T.$$  

Now suppose that $A \leq_m A^M_T$. We know that $A^M_T$ is Turing-recognizable, so by Theorem 5.28, $A$ is Turing-recognizable.

3. Problem 5.23

A is decidable if and only if $A \leq_m 0^*1^*$.

**Proof:** First suppose that $A$ is decidable. Define $f$ by $f(x) = 01$ if $x \in A$ and $f(x) = 10$ if $x \notin A$. Since $A$ is decidable, $f$ is computable and $x \in A$ if and only if $f(x) \in 0^*1^*$, so $A \leq_m 0^*1^*$.

Conversely, suppose that $A \leq_m 0^*1^*$. Since $0^*1^*$ is decidable, $A$ is decidable by Theorem 5.22.

4. Problem 5.24

The set $\overline{A^M_T}$ is mapping reduced to $J$ by the function $f(y) = 1y$. Thus, $J$ is not Turing-recognizable. The set $A^M_T$ is mapping reduced to $J$ by the function $g(x) = 0x$. This shows that $\overline{A^M_T}$ is mapping reducible to $J$ and hence that $J$ is not Turing-recognizable.

5. Problem 5.25
Consider the set $J$ of Problem 5.24. According to that problem, $J$ is not Turing-recognizable, so $J$ is not decidable. We will show that $J \leq_m B$, so $B = J$ is a solution to the problem.

First note that $J = \{w | w = 0x \text{ for some } x \in \overline{A_{TM}} \text{ or } w = 1y \text{ for some } y \in A_{TM} \text{ or } w = \varepsilon \text{ or } w \text{ begins with a symbol other than 0 or 1}\}$. Let $z_0$ be some fixed string in $J$, for example, $z_0$ could be $0x_0$ for some particular $x_0$ in $A_{TM}$. Define $f : \Sigma^* \rightarrow \Sigma^*$ by

$$f(w) = \begin{cases} 1x & \text{if } w = 0x \\ 0y & \text{if } w = 1y \\ z_0 & \text{if } w = \varepsilon \text{ or } w \text{ starts with a symbol other than 0 or 1.} \end{cases}$$

Then, it is clear that $f$ is computable. To see that $f$ mapping reduces $J$ to $J$, suppose first that $w \in J$. We must show that $f(w) \in J$. If $w \in J$, there are two possibilities. If $w = 0x$ with $x \in A_{TM}$, then $f(w) = 1x$ with $x \in A_{TM}$, so $f(w) \in J$. If $w = 1y$ with $y \in \overline{A_{TM}}$, then $f(w) = 0y$ with $y \in \overline{A_{TM}}$, so $f(w) \in \overline{J}$. Thus, if $w \in J$, then $f(w) \in J$.

Now suppose that $w \notin J$. We must show that $f(w) \notin J$. There are four possibilities to consider. If $w = 0x$ with $x \in \overline{A_{TM}}$, then $f(w) = 1x$, so $f(w) \notin J$. If $w = 1y$ with $y \in A_{TM}$, then $f(w) = 0y$, so $f(w) \notin J$. If $w = \varepsilon$ or $w$ starts with a symbol other than 0 or 1, then $f(w) = z_0$, so $f(w) \notin J$. Thus, if $w \notin J$, then $f(w) \notin J$.

This shows that $f$ is a mapping reduction of $J$ to $J$.

6. It is not possible to m-reduce $E_{LBA}$ to $A_{LBA}$.
   **Proof:** Suppose that $E_{LBA} \leq_m A_{LBA}$. By Theorem 5.9, $A_{LBA}$ is decidable, so by Theorem 5.22, $E_{LBA}$ is decidable. This contradicts Theorem 5.10, so the m-reduction is not possible.

7. Is $\overline{A_{LBA}}$ m-reducible to $0^*1^*$? Explain your answer.
   **Solution:** $\overline{A_{LBA}}$ is m-reducible to $0^*1^*$. To prove this, first note that by Theorem 5.9, $A_{LBA}$ is decidable, so by Problem 3.15d, $\overline{A_{LBA}}$ is decidable, so by Problem 5.23, $\overline{A_{LBA}}$ is m-reducible to $0^*1^*$.

8. Is $A_{TM}$ m-reducible to $\overline{REJECT_{TM}}$? Explain your answer.
   **Solution:** $A_{TM}$ is not m-reducible to $\overline{REJECT_{TM}}$. To prove this, suppose that $A_{TM} \leq_m \overline{REJECT_{TM}}$. Then $\overline{A_{TM}} \leq_m \overline{REJECT_{TM}}$. By Corollary 4.23, $\overline{A_{TM}}$ is not Turing recognizable, so by Corollary 5.29, $\overline{REJECT_{TM}}$ is not recognizable. This contradicts Problem 1a on Homework 10. Thus, the reduction is not possible.