1. Let $F = \{ \langle A \rangle | A$ is a DFA with input alphabet $\{0, 1\}$ and some string in $L(A)$ contains exactly three 1’s $\}$. Prove that $F$ is decidable.

2. Let $K = \{ \langle A \rangle | A$ is a DFA with alphabet $\{a, b\}$ and $L(A)$ does not contain any string with exactly one more $a$ than $b$ $\}$. Show that $K$ is decidable. (The solution to Problem 4.25 [4.23] is useful here.)

3. Let $L = \{ \langle P \rangle | P$ is a PDA with input alphabet $\{0, 1\}$ and no string in $L(P)$ contains exactly three 1’s $\}$. Prove that $L$ is decidable.

4. Show that the set that consists of all finite sequences of natural numbers is countably infinite. In other words show that

$$\{ s | \text{for some } n \geq 0 \text{ and natural numbers } a_1, \ldots, a_n, s = \langle a_1, \ldots, a_n \rangle \}$$

is a countably infinite set.

5. Let $A$ be a countably infinite set, $B$ be a set, and $f : A \to B$ be onto. Prove that $B$ is countable.

6. An infinite sequence of natural numbers $a(1)a(2)a(3) \cdots$ is called strictly increasing if $a(1) < a(2) < a(3) < \cdots$. Let $B$ be the set of all strictly increasing sequences of natural numbers. Use diagonalization to prove that $B$ is uncountable.

7. Let $C$ be the set of infinite binary sequences $a(1)a(2)a(3) \cdots$ such that $a(1) = a(3) = a(5) = \cdots = 0$. In other words a sequence in $C$ can have either 0 or 1 in the even positions, but has to have 0 in the odd positions. Prove that $C$ is uncountable.