Predicate Calculus

Universal Quantification

Let P(x) be a propositional function.

Universally quantified sentence: For all x in the universe of discourse P(x) is true.

Using the universal quantifier ∀:

∀x P(x) "for all x P(x)" or "for every x P(x)"

(Note: ∀x P(x) is either true or false, so it is a proposition, not a propositional function.)

Existential Quantification

Existentially quantified sentence: There exists an x in the universe of discourse for which P(x) is true.

Using the existential quantifier ∃:

∃x P(x) "There is an x such that P(x)."
"There is at least one x such that P(x)."

(Note: ∃x P(x) is either true or false, so it is a proposition, but no propositional function.)
Quantification

Another example:
Let the universe of discourse be the real numbers.

What does \( \forall x \exists y (x + y = 320) \) mean?

"For every \( x \) there exists a \( y \) so that \( x + y = 320 \)."

Is it true? yes

Is it true for the natural numbers? no

Disproof by Counterexample

A counterexample to \( \forall x \ P(x) \) is an object \( c \) so that \( P(c) \) is false.

Statements such as \( \forall x \ (P(x) \to Q(x)) \) can be disproved by simply providing a counterexample.

Statement: “All birds can fly.”
Disproved by counterexample: Penguin.

Negation

\( \neg (\forall x \ P(x)) \) is logically equivalent to \( \exists x \ (\neg P(x)) \).

\( \neg (\exists x \ P(x)) \) is logically equivalent to \( \forall x \ (\neg P(x)) \).

Quantification

Introducing the universal quantifier \( \forall \) and the existential quantifier \( \exists \) facilitates the translation of world knowledge into predicate calculus.

Examples:

Paul beats up all professors who fail him.

\( \forall x [(\text{Professor}(x) \land \text{Fails}(x, \text{Paul})) \to \text{BeatsUp}(\text{Paul}, x)] \)

All computer scientists are either rich or crazy, but not both.

\( \forall x (\text{CS}(x) \to (\text{Rich}(x) \land \neg \text{Crazy}(x)) \lor (\neg \text{Rich}(x) \land \text{Crazy}(x))) \)

Or, using XOR:

\( \forall x (\text{CS}(x) \to (\text{Rich}(x) \oplus \text{Crazy}(x))) \)

More Practice for Predicate Logic

Important points:

- Define propositional functions in a useful and reusable manner, just like functions in a computer program.

- Make sure your formalized statement evaluates to "true" in the context of the original statement and evaluates to "false" whenever the original statement is violated.

More Practice for Predicate Logic

More Examples:

Jenny likes all movies that Peter likes (and possibly more).

\( \forall \text{Movie}(x) \land \text{Likes}(\text{Peter}, x) \to \text{Likes}(\text{Jenny}, x) \)

There is exactly one UMass professor who won a Nobel prize

\( \exists x [(\text{UMProf}(x) \land \text{Wins}(x, \text{NobelPrize})) \land \neg \exists y, z [(y \neq z \land \text{UMProf}(y) \land \text{UMProf}(z) \land \text{Wins}(y, \text{NobelPrize}) \land \text{Wins}(z, \text{NobelPrize})]] \)
Rules of Inference for Quantified Statements

\[ \forall x \, P(x) \]

\[ \therefore \, P(c) \text{ if } c \in U \]

Universal instantiation

\[ P(c) \text{ for an arbitrary } c \in U \]

\[ \therefore \, \forall x \, P(x) \]

Universal generalization

\[ \exists x \, P(x) \]

\[ \therefore \, P(c) \text{ for some element } c \in U \]

Existential instantiation

\[ P(c) \text{ for some element } c \in U \]

\[ \therefore \, \exists x \, P(x) \]

Existential generalization

Rules of Inference for Quantified Statements

Example:

Every UMB student is a genius.
George is a UMB student.
Therefore, George is a genius.

U(x): “x is a UMB student.”
G(x): “x is a genius.”

The following steps are used in the argument:

Step 1: \( \forall x \, (U(x) \rightarrow G(x)) \) Hypothesis
Step 2: \( U(\text{George}) \rightarrow G(\text{George}) \) Univ. instantiation using Step 1
Step 3: \( U(\text{George}) \) Hypothesis
Step 4: \( G(\text{George}) \) Modus ponens using Steps 2 & 3

\[ \forall x \, P(x) \]

\[ \therefore \, P(c) \text{ if } c \in U \]

Universal instantiation