**Cartesian Product**

The ordered n-tuple \((a_1, a_2, a_3, \ldots, a_n)\) is an ordered collection of objects.

Two ordered n-tuples \((a_1, a_2, a_3, \ldots, a_n)\) and \((b_1, b_2, b_3, \ldots, b_n)\) are equal if and only if they contain exactly the same elements in the same order, i.e., \(a_i = b_i\) for \(1 \leq i \leq n\).

The Cartesian product of two sets is defined as:

\[A \times B = \{(a, b) | a \in A \land b \in B\}\]

Example: \(A = \{x, y\}, B = \{a, b, c\}\)

\[A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}\]

**Partitions**

**Definition:** A partition of a set \(S\) is a collection of disjoint nonempty subsets of \(S\) that have \(S\) as their union. In other words, the collection of subsets \(A_i, i \in I\), forms a partition of \(S\) if and only if

(i) \(A_i \neq \emptyset\) for \(i \in I\)

(ii) \(A_i \cap A_j = \emptyset\), if \(i \neq j\)

(iii) \(\cup_{i \in I} A_i = S\)

**Set Operations**

**Union:** \(A \cup B = \{x | x \in A \lor x \in B\}\)

Example: \(A = \{a, b\}, B = \{b, c, d\}\)

\[A \cup B = \{a, b, c, d\}\]

**Intersection:** \(A \cap B = \{x | x \in A \land x \in B\}\)

Example: \(A = \{a, b\}, B = \{b, c, d\}\)

\[A \cap B = \{b\}\]

**Notes:**

- \(A \times \emptyset = \emptyset\)
- \(\emptyset \times A = \emptyset\)
- For non-empty sets \(A\) and \(B\): \(A \neq B \iff A \times B \neq B \times A\)
- \(|A \times B| = |A| \cdot |B|\)

The Cartesian product of two or more sets is defined as:

\[A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } 1 \leq i \leq n\}\]
Set Operations
The complement of a set $A$ contains exactly those elements under consideration that are not in $A$:

$$-A = U-A$$

Example: $U = \mathbb{N}$, $B = \{250, 251, 252, \ldots\}$

$$-B = \{0, 1, 2, \ldots, 248, 249\}$$

Table 1 in Section 2.2 (4th edition: Section 1.5; 5th edition: Section 1.7; 6th edition: Section 2.2) shows many useful equations for set identities.

Set Operations
How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Method I:

$x \in A \cup (B \cap C) 
\iff x \in A \lor (x \in B \land x \in C) 
\iff (x \in A \lor x \in B) \land (x \in A \lor x \in C)$

(distributive law for logical expressions)

$x \in (A \cup B) \land (x \in A \lor x \in C) 
\iff x \in (A \cup B) \cap (A \cup C)$

Method II: Membership table
1 means "$x$ is an element of this set"
0 means "$x$ is not an element of this set"

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<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$B \cap C$</th>
<th>$A \cap (B \cap C)$</th>
<th>$A \cup B$</th>
<th>$A \cap (A \cup C)$</th>
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Set Operations
Take-home message:
Every logical expression can be transformed into an equivalent expression in set theory and vice versa.

Exercises

Question 1:
Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $|2^A \times 2^B|$?

Question 2:
Is it true for all sets $A$ and $B$ that $(A \times B) \cap (B \times A) = \emptyset$? Or do $A$ and $B$ have to meet certain conditions?

Question 3:
For any two sets $A$ and $B$, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$? Why or why not?

Question 1:
Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $|2^A \times 2^B|$?

Answer:

$|2^A \times 2^B| = |2^A| \cdot |2^B| = 2^{|A|} \cdot 2^{|B|} = 8 \cdot 16 = 128$
Exercises

Question 2:
Is it true for all sets A and B that \((A \times B) \cap (B \times A) = \emptyset\)?
Or do A and B have to meet certain conditions?

Answer:
If A and B share at least one element x, then both \((A \times B)\) and \((B \times A)\) contain the pair \((x, x)\) and thus are not disjoint.
Therefore, for the above equation to be true, it is necessary that \(A \cap B = \emptyset\).

Exercises

Question 3:
For any two sets A and B, if \(A - B = \emptyset\) and \(B - A = \emptyset\), can we conclude that \(A = B\)? Why or why not?

Answer:
Proof by contradiction: Assume that \(A \neq B\).
Then there must be either an element \(x\) such that \(x \in A\) and \(x \notin B\) or an element \(y\) such that \(y \in B\) and \(y \notin A\).
If \(x\) exists, then \(x \in (A - B)\), and thus \(A - B \neq \emptyset\).
If \(y\) exists, then \(y \in (B - A)\), and thus \(B - A \neq \emptyset\).
This contradicts the premise \(A - B = \emptyset\) and \(B - A = \emptyset\), and therefore we can conclude \(A = B\).

... and the next chapter is about...

Functions

A function \(f\) from a set \(A\) to a set \(B\) is an assignment of exactly one element of \(B\) to each element of \(A\).
We write \(f(a) = b\)
if \(b\) is the unique element of \(B\) assigned by the function \(f\) to the element \(a\) of \(A\).

If \(f\) is a function from \(A\) to \(B\), we write
\(f: A \rightarrow B\)
(note: Here, "\(\rightarrow\)" has nothing to do with if... then)

Functions

If \(f: A \rightarrow B\), we say that \(A\) is the domain of \(f\) and \(B\) is the codomain of \(f\).
If \(f(a) = b\), we say that \(b\) is the image of \(a\) and \(a\) is the pre-image of \(b\).
The range of \(f: A \rightarrow B\) is the set of all images of elements of \(A\).
We say that \(f: A \rightarrow B\) maps \(A\) to \(B\).
Functions

Let us re-specify $f$ as follows:

- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong Kong$
- $f(Peter) = Boston$

Is $f$ still a function? yes
What is its range? (Moscow, Boston, Hong Kong)

Other ways to represent $f$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
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</thead>
<tbody>
<tr>
<td>Linda</td>
<td>Moscow</td>
</tr>
<tr>
<td>Max</td>
<td>Boston</td>
</tr>
<tr>
<td>Kathy</td>
<td>Hong Kong</td>
</tr>
<tr>
<td>Peter</td>
<td>Moscow</td>
</tr>
</tbody>
</table>

If the domain of our function $f$ is large, it is convenient to specify $f$ with a formula, e.g.:

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x$

This leads to:

- $f(1) = 2$
- $f(3) = 6$
- $f(-3) = -6$
- ...

Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$. Then the sum and the product of $f_1$ and $f_2$ are also functions from $A$ to $\mathbb{R}$ defined by:

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1f_2)(x) = f_1(x) f_2(x)$

Example:

- $f_1(x) = 3x$, $f_2(x) = x + 5$
- $(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$
- $(f_1f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x$

We already know that the range of a function $f: A \to B$ is the set of all images of elements $a \in A$.

If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of $S$.

We denote the image of $S$ by $f(S)$:

$f(S) = \{f(s) \mid s \in S\}$

Let us look at the following well-known function:

- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong Kong$
- $f(Peter) = Boston$

What is the image of $S = \{Linda, Max\}$? $f(S) = \{Moscow, Boston\}$

What is the image of $S = \{Max, Peter\}$? $f(S) = \{Boston\}$
Properties of Functions

A function $f: A \rightarrow B$ is said to be one-to-one (or injective), if and only if

\[ \forall x, y \in A \ (f(x) = f(y) \implies x = y) \]

In other words: $f$ is one-to-one if and only if it does not map two distinct elements of $A$ onto the same element of $B$.

And again…

- $f(\text{Linda}) = \text{Moscow}$
- $f(\text{Max}) = \text{Boston}$
- $f(\text{Kathy}) = \text{Hong Kong}$
- $f(\text{Peter}) = \text{Boston}$

Is $f$ one-to-one?

No, Max and Peter are mapped onto the same element of the image.

Properties of Functions

- $f(\text{Linda}) = \text{Moscow}$
- $f(\text{Max}) = \text{Boston}$
- $f(\text{Kathy}) = \text{Hong Kong}$
- $f(\text{Peter}) = \text{Boston}$

- $g(\text{Linda}) = \text{Moscow}$
- $g(\text{Max}) = \text{Boston}$
- $g(\text{Kathy}) = \text{Hong Kong}$
- $g(\text{Peter}) = \text{New York}$

Is $g$ one-to-one?

Yes, each element is assigned a unique element of the image.