Homework Solution

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<tr>
<th>P</th>
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<th>¬(P ∨ Q)</th>
<th>(¬P) ∧ (¬Q)</th>
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The statements ¬(P ∨ Q) and (¬P) ∧ (¬Q) are logically equivalent, so we write ¬(P ∨ Q) ⇔ (¬P) ∧ (¬Q).

... and now for something completely different...

Set Theory

Actually, you will see that logic and set theory are very closely related.

Set Theory

• Set: Collection of objects ("elements")
  • a ∈ A "a is an element of A"
  • a ∉ A "a is not an element of A"
  • A = {a₁, a₂, …, aₙ} "A contains…"
  • Order of elements is meaningless
  • It does not matter how often the same element is listed.

Set Equality

Sets A and B are equal if and only if they contain exactly the same elements.

Examples:
  • A = {9, 2, 7, -3}, B = {7, 9, -3, 2} : A = B
  • A = {dog, cat, horse}, B = {cat, horse, squirrel, dog} : A ≠ B
  • A = {dog, cat, horse}, B = {cat, horse, dog, dog} : A = B

Examples for Sets

"Standard" Sets:
  • Natural numbers N = {0, 1, 2, 3, …}
  • Integers Z = {…, -2, -1, 0, 1, 2, …}
  • Positive Integers Z⁺ = {1, 2, 3, 4, …}
  • Real Numbers R = {47.3, -12, π, …}
  • Rational Numbers Q = {1.5, 2.6, -3.8, 15, …} (correct definition will follow)
Examples for Sets

We are now able to define the set of rational numbers $\mathbb{Q}$:

$$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \land b \in \mathbb{Z}^+\}$$

or

$$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \land b \in \mathbb{Z} \land b \neq 0\}$$

And how about the set of real numbers $\mathbb{R}$?

$$\mathbb{R} = \{r \mid r \text{ is a real number}\}$$

That is the best we can do.

Subsets

A $\subseteq B$ “A is a subset of B”

A $\subseteq B$ if and only if every element of A is also an element of B.

We can completely formalize this:

$$A \subseteq B \iff \forall x (x \in A \rightarrow x \in B)$$

Examples:

- $A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true
- $A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true
- $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? false

Useful rules:

- $\emptyset \subseteq A$ for any set A
- $A \subseteq A$ for any set A

Proper subsets:

A $\subset B$ “A is a proper subset of B”

$$A \subset B \iff \forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$$

or

$$A \subset B \iff \forall x (x \in A \rightarrow x \in B) \land \neg \forall x (x \in B \rightarrow x \in A)$$

Cardinality of Sets

If a set S contains n distinct elements, $n \in \mathbb{N}$, we call S a finite set with cardinality n.

Examples:

- $A = \{\text{Mercedes, BMW, Porsche}\}$, $|A| = 3$
- $B = \{1, 2, 3, 4, 5, 6\}$, $|B| = 4$
- $C = \emptyset$, $|C| = 0$
- $D = \{x \in \mathbb{N} \mid x \leq 7000\}$, $|D| = 7001$
- $E = \{x \in \mathbb{N} \mid x \geq 7000\}$, E is infinite!

The Power Set

$2^A$ or $P(A)$ “power set of A”

$2^A = \{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

- $A = \{x, y, z\}$
- $2^A = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$
- $A = \emptyset$
- $2^A = \{\emptyset\}$
- Note: $|A| = 0$, $|2^A| = 1$
The Power Set
Cardinality of power sets:
\[ |2^A| = 2^{|A|} \]
- Imagine each element in A has an “on/off” switch
- Each possible switch configuration in A corresponds to one element in \(2^A\)

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<tr>
<th>A</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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- For 3 elements in A, there are \(2 \times 2 \times 2 = 8\) elements in \(2^A\)

Cartesian Product
The ordered n-tuple \((a_1, a_2, a_3, \ldots, a_n)\) is an ordered collection of objects.
Two ordered n-tuples \((a_1, a_2, a_3, \ldots, a_n)\) and \((b_1, b_2, b_3, \ldots, b_n)\) are equal if and only if they contain exactly the same elements in the same order, i.e., \(a_i = b_i\) for \(1 \leq i \leq n\).

The Cartesian product of two sets is defined as:
\[ A \times B = \{(a, b) | a \in A \land b \in B\} \]
Example: \(A = \{x, y\}, B = \{a, b, c\}\)
\[ A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\} \]

Note that:
- \(A \times \emptyset = \emptyset\)
- \(\emptyset \times A = \emptyset\)
- For non-empty sets A and B: \(A \times B \leftrightarrow A \times B \neq B \times A\)
- \(|A \times B| = |A| \cdot |B|\)

The Cartesian product of two or more sets is defined as:
\[ A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } 1 \leq i \leq n\} \]

Set Operations
Two sets are called disjoint if their intersection is empty, that is, they share no elements:
\(A \cap B = \emptyset\)

The difference between two sets A and B contains exactly those elements of A that are not in B:
\(A-B = \{x | x \in A \land x \notin B\}\)
Example: \(A = \{a, b\}, B = \{b, c, d\}\)
\(A-B = \{a\}\)

Set Operations
The complement of a set A contains exactly those elements under consideration that are not in A:
\(-A = U-A\)
Example: \(U = \mathbb{N}, B = \{250, 251, 252, \ldots\}\)
\(-B = \{0, 1, 2, \ldots, 248, 249\}\)

Table 1 in Section 2.2 (4th edition: Section 1.5; 5th edition: Section 1.7; 6th edition: Section 2.2) shows many useful equations for set identities.
Set Operations

How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Method I:

$x \in A \cup (B \cap C)$

$\iff x \in A \lor x \in (B \cap C)$

$\iff x \in A \lor (x \in B \land x \in C)$

$\iff (x \in A \lor x \in B) \land (x \in A \lor x \in C)$

(distributive law for logical expressions)

$\iff x \in (A \cup B) \land x \in (A \cup C)$

$\iff x \in (A \cup B) \cap (A \cup C)$

Set Operations

Take-home message:

Every logical expression can be transformed into an equivalent expression in set theory and vice versa.

Set Operations

Method II: Membership table

1 means "x is an element of this set"

0 means "x is not an element of this set"

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Exercises

Question 1:

Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $|2^A \times 2^B|$?

Question 2:

Is it true for all sets $A$ and $B$ that $(A \times B) \cap (B \times A) = \emptyset$?

Or do $A$ and $B$ have to meet certain conditions?

Question 3:

For any two sets $A$ and $B$, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$? Why or why not?