Permutations and Combinations

**Corollary:**
Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \).
Then \( C(n, r) = C(n, n - r) \).

Note that “picking a group of \( r \) people from a group of \( n \) people” is the same as “splitting a group of \( n \) people into a group of \( r \) people and another group of \( n - r \) people”.

**Combinations**

**Proof:**
\[
C(n, n-r) = \frac{n!}{(n-r)![(n-(n-r))]!} = \frac{n!}{(n-r)!r!} = C(n, r)
\]

This symmetry is intuitively plausible. For example, let us consider a set containing six elements \( (n = 6) \).

**Picking two elements and leaving four** is essentially the same as **picking four elements and leaving two**.

In either case, our number of choices is the number of possibilities to divide the set into one set containing two elements and another set containing four elements.

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**Example:**
A soccer club has 8 female and 7 male members. For today’s match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

\[
C(8, 6) \cdot C(7, 5) = \frac{8!}{(6!2!)} \cdot \frac{7!}{(5!2!)} = 28 \cdot 21 = 588
\]

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**Pascal’s Identity:**
Let \( n \) and \( k \) be positive integers with \( n \geq k \).
Then \( C(n + 1, k) = C(n, k - 1) + C(n, k) \).

How can this be explained?
What is it good for?

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**Pascal’s Triangle**
In Pascal’s triangle, each number is the sum of the numbers to its upper left and upper right:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]
Pascal's Triangle
Since we have $C(n + 1, k) = C(n, k – 1) + C(n, k)$ and $C(0, 0) = 1$, we can use Pascal's triangle to simplify the computation of $C(n, k)$:

<table>
<thead>
<tr>
<th>k</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1  2</td>
</tr>
<tr>
<td>3</td>
<td>1  3  3  1</td>
</tr>
<tr>
<td>4</td>
<td>1  4  6  4  1</td>
</tr>
</tbody>
</table>

Binomial Coefficients
Expressions of the form $C(n, k)$ are also called **binomial coefficients**.

How come?
A binomial expression is the sum of two terms, such as $(a + b)$.
Now consider $(a + b)^2 = (a + b)(a + b)$.
When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:

$$(a + b)^2 = a \cdot a + a \cdot b + b \cdot a + b \cdot b$$

Then we can sum identical terms:

$$(a + b)^2 = a^2 + 2ab + b^2$$

This leads us to the following formula:

$$(a + b)^n = \sum_{j=0}^{n} C(n, j) \cdot a^{n-j} \cdot b^j \quad \text{(Binomial Theorem)}$$

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle $(1 – 4 – 6 – 4 – 1)$ helps us to compute $(a + b)^4$:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Discrete Probability
Everything you have learned about counting constitutes the basis for computing the **probability** of events to happen.

In the following, we will use the notion **experiment** for a procedure that yields one of a given set of possible outcomes.

This set of possible outcomes is called the **sample space** of the experiment.

An **event** is a subset of the sample space.

Now it's time to look at…

Discrete Probability
Discrete Probability
If all outcomes in the sample space are equally likely, the following definition of probability applies:

The probability of an event E, which is a subset of a finite sample space S of equally likely outcomes, is given by \( p(E) = \frac{|E|}{|S|} \).

Probability values range from 0 (for an event that will never happen) to 1 (for an event that will always happen whenever the experiment is carried out).

Discrete Probability

Example I:
An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

Solution:
There are nine possible outcomes, and the event “blue ball is chosen” comprises four of these outcomes. Therefore, the probability of this event is 4/9 or approximately 44.44%.

Discrete Probability

Example II:
What is the probability of winning the lottery 6/49, that is, picking the correct set of six numbers out of 49?

Solution:
There are \( \binom{49}{6} \) possible outcomes. Only one of these outcomes will actually make us win the lottery.

\[ p(E) = \frac{1}{\binom{49}{6}} = \frac{1}{13,983,816} \]

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Complementary Events
Let E be an event in a sample space S. The probability of an event \( \neg E \), the complementary event of E, is given by

\[ p(\neg E) = 1 - p(E) \]

This can easily be shown:

\[ p(\neg E) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E) \]

This rule is useful if it is easier to determine the probability of the complementary event than the probability of the event itself.

Complementary Events

Example I: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is zero?

Solution: There are \( 2^{10} = 1024 \) possible outcomes of generating such a sequence. The event \( \neg E \), “none of the bits is zero”, includes only one of these outcomes, namely the sequence 1111111111.

Therefore, \( p(\neg E) = 1/1024 \).

Now \( p(E) \) can easily be computed as

\[ p(E) = 1 - p(\neg E) = 1 - 1/1024 = 1023/1024 \]
Complementary Events

**Example II:** What is the probability that at least two out of 36 people have the same birthday?

**Solution:** The sample space $S$ encompasses all possibilities for the birthdays of the 36 people, so $|S| = 365^{36}$.

Let us consider the event $-E$ ("no two people out of 36 have the same birthday"). $-E$ includes $P(365, 36)$ outcomes (365 possibilities for the first person’s birthday, 364 for the second, and so on). Then $p(-E) = P(365, 36)/365^{36} = 0.168$, so $p(E) = 0.832$ or 83.2%.

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Discrete Probability

Let $E_1$ and $E_2$ be events in the sample space $S$. Then we have:

$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Does this remind you of something?

Of course, the principle of inclusion-exclusion.