Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.

Since this is a relation between **two sets**, it is called a **binary relation**.

**Definition:** Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, for a binary relation $R$ we have $R \subseteq A \times B$. We use the notation $aRb$ to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \not\in R$.

**Relations**

When $(a, b)$ belongs to $R$, $a$ is said to be related to $b$ by $R$.

**Example:** Let $P$ be a set of people, $C$ be a set of cars, and $D$ be the relation describing which person drives which car(s).

$P = \{\text{Carl, Suzanne, Peter, Carla}\}$,

$C = \{\text{Mercedes, BMW, tricycle}\}$

$D = \{(\text{Carl, Mercedes}), (\text{Suzanne, Mercedes}), (\text{Suzanne, BMW}), (\text{Peter, tricycle})\}$

This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

**Functions as Relations**

You might remember that a function $f$ from a set $A$ to a set $B$ assigns a unique element of $B$ to each element of $A$.

The graph of $f$ is the set of ordered pairs $(a, b)$ such that $b = f(a)$.

Since the graph of $f$ is a subset of $A \times B$, it is a relation from $A$ to $B$.

Moreover, for each element $a$ of $A$, there is exactly one ordered pair in the graph that has $a$ as its first element.

**Relations on a Set**

**Definition:** A relation on the set $A$ is a relation from $A$ to $A$.

In other words, a relation on the set $A$ is a subset of $A \times A$.

**Example:** Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?
Relations on a Set

Solution: \( R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \)

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation \( R \) on a set \( A \) is called reflexive if \((a, a) \in R\) for every element \( a \in A \).

Are the following relations on \( \{1, 2, 3, 4\} \) reflexive?

- \( R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\} \) Yes.
- \( R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\} \) No.

Definition: A relation \( R \) on a set \( A \) is called irreflexive if \((a, a) \not\in R\) for every element \( a \in A \).

Properties of Relations

Definitions:

A relation \( R \) on a set \( A \) is called symmetric if \((b, a) \in R \) whenever \((a, b) \in R\) for all \( a, b \in A \).

A relation \( R \) on a set \( A \) is called antisymmetric if \( a = b \) whenever \((a, b) \in R \) and \((b, a) \in R\).

A relation \( R \) on a set \( A \) is called asymmetric if \((a, b) \in R\) implies that \((b, a) \not\in R\) for all \( a, b \in A \).

Counting Relations

Example: How many different reflexive relations can be defined on a set \( A \) containing \( n \) elements?

Solution: Relations on \( R \) are subsets of \( A \times A \), which contains \( n^2 \) elements.

Therefore, different relations on \( A \) can be generated by choosing different subsets out of these \( n^2 \) elements, so there are \( 2^{n^2} \) relations.

A reflexive relation, however, must contain the \( n \) elements \((a, a)\) for every \( a \in A \).

Consequently, we can only choose among \( n^2 - n = n(n-1) \) elements to generate reflexive relations, so there are \( 2^{n(n-1)} \) of them.
Combining Relations

Relations are sets, and therefore, we can apply the usual set operations to them.

If we have two relations $R_1$ and $R_2$, and both of them are from a set $A$ to a set $B$, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

In each case, the result will be another relation from $A$ to $B$.

Combining Relations

... and there is another important way to combine relations.

Definition: Let $R$ be a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The composite of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

In other words, if relation $R$ contains a pair $(a, b)$ and relation $S$ contains a pair $(b, c)$, then $S \circ R$ contains a pair $(a, c)$.

Combining Relations

Example: Let $D$ and $S$ be relations on $A = \{1, 2, 3, 4\}$.

$D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"

$S = \{(a, b) \mid a < b\}$ "a is smaller than b"

$D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}

$S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}

$S \circ D = \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}

$D$ maps an element $a$ to the element $(5 – a)$, and afterwards $S$ maps $(5 – a)$ to all elements larger than $(5 – a)$, resulting in $S \circ D = \{(a, b) \mid b > 5 – a\}$ or $S \circ D = \{(a, b) \mid a + b > 5\}$.