Equivalence Relations

**Equivalence relations** are used to relate objects that are similar in some way.

**Definition:** A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation $R$ are called **equivalent**.

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Equivalence Relations

Since $R$ is **symmetric**, $a$ is equivalent to $b$ whenever $b$ is equivalent to $a$.

Since $R$ is **reflexive**, every element is equivalent to itself.

Since $R$ is **transitive**, if $a$ and $b$ are equivalent and $b$ and $c$ are equivalent, then $a$ and $c$ are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

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Equivalence Classes

**Example:** Suppose that $R$ is the relation on the set of strings that consist of English letters such that $aRb$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string $x$. Is $R$ an equivalence relation?

**Solution:**

- $R$ is reflexive, because $l(a) = l(a)$ and therefore $aRa$ for any string $a$.
- $R$ is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if $aRb$ then $bRa$.
- $R$ is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so $aRb$ and $bRc$ implies $aRc$.

$R$ is an equivalence relation.

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Equivalence Classes

**Definition:** Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the **equivalence class** of $a$.

The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$.

When only one relation is under consideration, we will delete the subscript $R$ and write $[a]$ for this equivalence class.

If $b \in [a]_R$, $b$ is called a **representative** of this equivalence class.

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Equivalence Classes

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

(i) $aRb$

(ii) $[a] = [b]$  

(iii) $[a] \cap [b] \neq \emptyset$

**Definition:** A partition of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union. In other words, the collection of subsets $A_i$, $i \in I$, forms a partition of $S$ if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$

(iii) $\bigcup_{i \in I} A_i = S$
### Equivalence Classes

**Examples:** Let $S$ be the set $\{u, m, b, r, o, c, k, s\}$. Do the following collections of sets partition $S$?

- $\{\{m, o, c, k\}, \{r, u, b, s\}\}$: yes.
- $\{\{c, o, m, b\}, \{u, s\}, \{r\}\}$: no (k is missing).
- $\{\{b, r, o, c, k\}, \{m, u, s, t\}\}$: no (t is not in S).
- $\{\{u, m, b, r, o, c, k, s\}\}$: yes.
- $\{\{b, o, r, k\}, \{r, u, m\}, \{c, s\}\}$: no (r is in two sets).
- $\{\{u, m, b\}, \{r, o, c, k, s\}, \emptyset\}$: no ($\emptyset$ not allowed).

### Equivalence Classes

**Theorem:** Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i | i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i, i \in I$, as its equivalence classes.

### Equivalence Classes

**Example:** Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney. Let $R$ be the equivalence relation $\{(a, b) | a$ and $b$ live in the same city$\}$ on the set $P = \{Frank, Suzanne, George, Stephanie, Max, Jennifer\}$. Then $R = \{(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)\}$.

Then the equivalence classes of $R$ are: $\{\{Frank, Suzanne, George\}, \{Stephanie, Max\}, \{Jennifer\}\}$. This is a partition of $P$.

The equivalence classes of any equivalence relation $R$ defined on a set $S$ constitute a partition of $S$, because every element in $S$ is assigned to exactly one of the equivalence classes.

### Equivalence Classes

**Another example:** Let $R$ be the relation $\{(a, b) | a = b \pmod{3}\}$ on the set of integers. Is $R$ an equivalence relation?

Yes, $R$ is reflexive, symmetric, and transitive.

What are the equivalence classes of $R$?

- $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$
- $\{\ldots, -5, -2, 1, 4, 7, \ldots\}$
- $\{\ldots, -4, -1, 2, 5, 8, \ldots\}$

Again, these three classes form a partition of the set of integers.

### Partial Orderings

Sometimes, relations do not specify the equality of elements in a set, but define an order on them.

**Definition:** A relation $R$ on a set $S$ is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.

A set $S$ together with a partial ordering $R$ is called a partially ordered set, or poset, and is denoted by $(S, R)$. 
Partial Orderings

**Example:** Consider the “greater than or equal” relation $\geq$ (defined by $\{(a, b) \mid a \geq b\}$). Is $\geq$ a *partial ordering* on the set of integers?

• $\geq$ is **reflexive**, because $a \geq a$ for every integer $a$.
• $\geq$ is **antisymmetric**, because if $a \neq b$, then $a \geq b \land b \geq a$ is false.
• $\geq$ is **transitive**, because if $a \geq b$ and $b \geq c$, then $a \geq c$.

Consequently, $(Z, \geq)$ is a partially ordered set.

Partial Orderings

Another example: Is the “inclusion relation” $\subseteq$ a *partial ordering* on the power set of a set $S$?

• $\subseteq$ is **reflexive**, because $A \subseteq A$ for every set $A$.
• $\subseteq$ is **antisymmetric**, because if $A \neq B$, then $A \subseteq B \land B \subseteq A$ is false.
• $\subseteq$ is **transitive**, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Consequently, $(P(S), \subseteq)$ is a partially ordered set.

Partial Orderings

In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.

Note that the symbol $\leq$ is used to denote the relation in *any* poset, not just the “less than or equal” relation. The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.

If $a < b$ we say “$a$ is less than $b$” or “$b$ is greater than $a$”.

Partial Orderings

**Example I:** Is $(Z, \leq)$ a totally ordered set? Yes, because $a \leq b$ or $b \leq a$ for all integers $a$ and $b$.

**Example II:** Is $(Z^+, |)$ a totally ordered set? No, because it contains incomparable elements such as 5 and 7.

Partial Orderings

For some applications, we require all elements of a set to be comparable.

For example, if we want to write a dictionary, we need to define an order on all English words (alphabetical order).

**Definition:** If $(S, \leq)$ is a poset and every two elements of $S$ are comparable, $S$ is called a *totally ordered* or *linearly ordered set*, and $\leq$ is called a *total order* or *linear order*. A totally ordered set is also called a *chain*.
Let us switch to a new topic:

**Graphs**

**Introduction to Graphs**

**Definition:** A simple graph $G = (V, E)$ consists of $V$, a nonempty set of vertices, and $E$, a set of unordered pairs of distinct elements of $V$ called edges.

A simple graph is just like a directed graph, but with no specified direction of its edges.

Sometimes we want to model multiple connections between vertices, which is impossible using simple graphs.

In these cases, we have to use multigraphs.

**Definition:** A multigraph $G = (V, E)$ consists of a nonempty set of vertices, $V$, and $E$, a set of unordered pairs of distinct elements of $V$ called edges.

A simple graph is just like a directed graph, but with no specified direction of its edges.

Sometimes we want to model multiple connections between vertices, which is impossible using simple graphs.

In these cases, we have to use multigraphs.

**Example:** A multigraph $G$ with vertices $V = \{a, b, c, d\}$, edges $\{1, 2, 3, 4, 5\}$ and function $f$ with $f(1) = \{a, b\}$, $f(2) = \{a, b\}$, $f(3) = \{b, c\}$, $f(4) = \{c, d\}$ and $f(5) = \{c, d\}$:

```
   a -- 1 -- b
     |   ^ 3
     |   |  c
     v   v
   2 -- 4 -- 5
```

**Note:**
- Edges in multigraphs are not necessarily defined as pairs, but can be of any type.
- No loops are allowed in multigraphs ($u \neq v$).

**Introduction to Graphs**

If we want to define loops, we need the following type of graph:

**Definition:** A pseudograph $G = (V, E)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f$ from $E$ to $\{(u, v) | u, v \in V \}$.

An edge $e$ is a loop if $f(e) = \{u, u\}$ for some $u \in V$.

**Definition:** A pseudograph $G = (V, E)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f$ from $E$ to $\{(u, v) | u, v \in V \}$.

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**Definition:** A pseudograph $G = (V, E)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f$ from $E$ to $\{(u, v) | u, v \in V \}$.

A directed graph $G = (V, E)$ consists of a set $V$ of vertices and a set $E$ of edges that are ordered pairs of elements in $V$.

...leading to a new type of graph:

**Definition:** A directed multigraph $G = (V, E)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f$ from $E$ to $\{(u, v) | u, v \in V \}$.

The edges $e_1$ and $e_2$ are called multiple edges if $f(e_1) = f(e_2)$. 
Examples: A directed multigraph $G$ with vertices $V = \{a, b, c, d\}$, edges $\{1, 2, 3, 4, 5\}$ and function $f$ with $f(1) = (a, b)$, $f(2) = (b, a)$, $f(3) = (c, b)$, $f(4) = (c, d)$ and $f(5) = (c, d)$:

```
1  2  3  4  5
|   |   |   |   |
a b c  d
```

Types of Graphs and Their Properties

| Type            | Edges | Multiple Edges? | Loops? 
<table>
<thead>
<tr>
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<td>yes</td>
</tr>
<tr>
<td>dir. multigraph</td>
<td>directed</td>
<td>yes</td>
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Example I: How can we represent a network of (bi-directional) railways connecting a set of cities? We should use a simple graph with an edge $\{a, b\}$ indicating a direct train connection between cities $a$ and $b$.

```
New York  Toronto  Chicago  Lübeck  Washington
|   |    |        |        |       |
|   |    |        |        |       |
|   |    |        |        |       |
|   |    |        |        |       |
```

Example II: In a round-robin tournament, each team plays against each other team exactly once. How can we represent the results of the tournament (which team beats which other team)? We should use a directed graph with an edge $(a, b)$ indicating that team $a$ beats team $b$.

```
Bruins  Penguins  Maple Leafs  Lübeck Giants
|   |        |          |         |
|   |        |          |         |
|   |        |          |         |
|   |        |          |         |
```

Definition: Two vertices $u$ and $v$ in an undirected graph $G$ are called adjacent (or neighbors) in $G$ if $\{u, v\}$ is an edge in $G$.

If $e = \{u, v\}$, the edge $e$ is called incident with the vertices $u$ and $v$. The edge $e$ is also said to connect $u$ and $v$.

The vertices $u$ and $v$ are called endpoints of the edge $\{u, v\}$.

Definition: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

In other words, you can determine the degree of a vertex in a displayed graph by counting the lines that touch it.

The degree of the vertex $v$ is denoted by $\deg(v)$.
Graph Terminology

A vertex of degree 0 is called isolated, since it is not adjacent to any vertex.

**Note:** A vertex with a loop at it has at least degree 2 and, by definition, is not isolated, even if it is not adjacent to any other vertex.

A vertex of degree 1 is called pendant. It is adjacent to exactly one other vertex.

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Example: Which vertices in the following graph are isolated, which are pendant, and what is the maximum degree? What type of graph is it?

Solution: Vertex f is isolated, and vertices a, d and j are pendant. The maximum degree is $\deg(g) = 5$. This graph is a pseudograph (undirected, loops).

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Graph Terminology

Let us look at the same graph again and determine the number of its edges and the sum of the degrees of all its vertices:

Result: There are 9 edges, and the sum of all degrees is 18. This is easy to explain: Each new edge increases the sum of degrees by exactly two.

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Graph Terminology

**The Handshaking Theorem:** Let $G = (V, E)$ be an undirected graph with $e$ edges. Then

$$2e = \sum_{v \in V} \deg(v)$$

**Note:** This theorem holds even if multiple edges and/or loops are present.

**Example:** How many edges are there in a graph with 10 vertices, each of degree 6?

**Solution:** The sum of the degrees of the vertices is $6 \cdot 10 = 60$. According to the Handshaking Theorem, it follows that $2e = 60$, so there are 30 edges.