Graph Theorems

**Theorem:** An undirected graph has an even number of vertices of odd degree.

**Idea:** There are three possibilities for adding an edge to connect two vertices in the graph:

**Before:**
- Both vertices have even degree
- Both vertices have odd degree
- One vertex has odd degree, the other even

**After:**
- Both vertices have odd degree
- Both vertices have even degree
- One vertex has even degree, the other odd

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Graph Theorems

There are two possibilities for adding a loop to a vertex in the graph:

**Before:**
- The vertex has even degree
- The vertex has odd degree

**After:**
- The vertex has even degree
- The vertex has odd degree

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Graph Terminology

So if there is an even number of vertices of odd degree in the graph, it will still be even after adding an edge.

Therefore, since an undirected graph with no edges has an even number of vertices with odd degree (zero), the same must be true for any undirected graph.

Please also study the proof on
- 4th Edition: page 446
- 5th Edition: page 547
- 6th edition: page 599
- 7th edition: page 653

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Graph Terminology

**Definition:** When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v, and v is said to be adjacent from u.

The vertex u is called the initial vertex of (u, v), and v is called the terminal vertex of (u, v).

The initial vertex and terminal vertex of a loop are the same.

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Graph Terminology

**Definition:** In a graph with directed edges, the in-degree of a vertex v, denoted by deg(v), is the number of edges with v as their terminal vertex. The out-degree of v, denoted by deg*(v), is the number of edges with v as their initial vertex.

**Question:** How does adding a loop to a vertex change the in-degree and out-degree of that vertex?

**Answer:** It increases both the in-degree and the out-degree by one.

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Graph Terminology

**Example:** What are the in-degrees and out-degrees of the vertices a, b, c, d in this graph:

- deg(a) = 1, deg*(a) = 2
- deg(b) = 4, deg*(b) = 2
- deg(d) = 2, deg*(d) = 1
- deg(c) = 0, deg*(c) = 2
Graph Terminology

Theorem: Let \( G = (V, E) \) be a graph with directed edges. Then:
\[
\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|
\]
This is easy to see, because every new edge increases both the sum of in-degrees and the sum of out-degrees by one.

Special Graphs

Definition: The complete graph on \( n \) vertices, denoted by \( K_n \), is the simple graph that contains exactly one edge between each pair of distinct vertices.

\[ K_1 \quad K_2 \quad K_3 \quad K_4 \quad K_5 \]

Special Graphs

Definition: The cycle \( C_n \), \( n \geq 3 \), consists of \( n \) vertices \( v_1, v_2, \ldots, v_n \) and edges \( \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\} \).

\[ C_3 \quad C_4 \quad C_5 \quad C_6 \]

Special Graphs

Definition: We obtain the wheel \( W_n \) when we add an additional vertex to the cycle \( C_n \), for \( n \geq 3 \), and connect this new vertex to each of the \( n \) vertices in \( C_n \) by adding new edges.

\[ W_3 \quad W_4 \quad W_5 \quad W_6 \]

Special Graphs

Definition: The \( n \)-cube, denoted by \( Q_n \), is the graph that has vertices representing the \( 2^n \) bit strings of length \( n \). Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

\[ Q_1 \quad Q_2 \quad Q_3 \]

Definition: A simple graph is called bipartite if its vertex set \( V \) can be partitioned into two disjoint nonempty sets \( V_1 \) and \( V_2 \) such that every edge in the graph connects a vertex in \( V_1 \) with a vertex in \( V_2 \) (so that no edge in \( G \) connects either two vertices in \( V_1 \) or two vertices in \( V_2 \)).

For example, consider a graph that represents each person in a mixed-doubles tennis tournament (i.e., teams consist of one female and one male player). Players of the same team are connected by edges.

This graph is bipartite, because each edge connects a vertex in the subset of males with a vertex in the subset of females.
Special Graphs

Example I: Is $C_3$ bipartite?
No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

Example II: Is $C_6$ bipartite?
Yes, because we can display $C_6$ like this:

Operations on Graphs

Definition: A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

Note: Of course, $H$ is a valid graph, so we cannot remove any endpoints of remaining edges when creating $H$.

Example:

Representing Graphs

Definition: Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose that the vertices of $G$ are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.

The adjacency matrix $A$ (or $A_G$) of $G$, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its $(i, j)$th entry when $v_i$ and $v_j$ are adjacent, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,$$a_{ij} = 1 \quad \text{if } v_i \text{ and } v_j \text{ are adjacent,}$$$$a_{ij} = 0 \quad \text{otherwise.}$$
Representing Graphs

Example: What is the adjacency matrix \( A_G \) for the following graph \( G \) based on the order of vertices \( a, b, c, d \)?

Solution: \[
A_G = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

Note: Adjacency matrices of undirected graphs are always symmetric.

For the representation of graphs with multiple edges, we can no longer use zero-one matrices. Instead, we use matrices of natural numbers. The \((i, j)\)th entry of such a matrix equals the number of edges that are associated with \( \{v_i, v_j\} \).

Definition: Let \( G = (V, E) \) be a directed graph with \(|V| = n\). Suppose that the vertices of \( G \) are listed in arbitrary order as \( v_1, v_2, \ldots, v_n \).

The adjacency matrix \( A \) (or \( A_d \)) of \( G \), with respect to this listing of the vertices, is the \( n \times n \) zero-one matrix with 1 as its \((i, j)\)th entry when there is an edge from \( v_i \) to \( v_j \), and 0 otherwise.

In other words, for an adjacency matrix \( A = [a_{ij}] \),
\[
a_{ij} = 1 \quad \text{if } (v_i, v_j) \text{ is an edge of } G,
\]
\[
a_{ij} = 0 \quad \text{otherwise}.
\]