Assignment #2

Sample Solutions

Note: This is a fillable PDF, and you need to put all of your answers into the answer boxes. Press CTRL + E to add formatting, and you can copy and paste symbols from this or other documents (like these: ∀ ∃ → ↔ ≤ ≥ × · ≠ ∩ ∪ ⊆ ⊇ ∈ ∉ −). You can do this digitally, without any paper involved, or you can print this assignment, fill in the answers with a pen, and then scan the result. The copiers in the department or in McCormack can scan and email you the scanned pages. Alternatively, there are scanning apps for all types of smart phones.

Afterwards, upload your answers to gradescope.com. If you have not signed up yet, please do so using code 9D5E5J.

Question 1: Prime Factorization

Write down the prime factorization (in ascending order) of each of the following integers (Example: 720 = 2^4 · 3^2 · 5).

a) 264

\[ 2^3 \cdot 3 \cdot 11 \]

b) 1000000

\[ 2^6 \cdot 5^6 \]

c) 2430

\[ 2 \cdot 3^5 \cdot 5 \]

d) 117

\[ 3^2 \cdot 13 \]

Question 2: Euclidean Algorithm

Use the Euclidean algorithm to determine the following greatest common divisors. Write down every step in your calculation.
a) \( \text{gcd}(3330, 550) \)

\[
\begin{align*}
3330/550 &= 6 \text{ R} \ 30 \Rightarrow \text{gcd}(550, 30) \\
550/30 &= 18 \text{ R} \ 10 \Rightarrow \text{gcd}(30, 10) \\
30/10 &= 3 \text{ R} \ 0 \Rightarrow \ 10
\end{align*}
\]

b) \( \text{gcd}(178, 300) \)

\[
\begin{align*}
300/178 &= 1 \text{ R} \ 122 \Rightarrow \text{gcd}(178, 122) \\
178/122 &= 1 \text{ R} \ 56 \Rightarrow \text{gcd}(122, 56) \\
122/56 &= 2 \text{ R} \ 10 \Rightarrow \text{gcd}(56, 10) \\
56/10 &= 5 \text{ R} \ 6 \Rightarrow \text{gcd}(10, 6) \\
10/6 &= 1 \text{ R} \ 4 \Rightarrow \text{gcd}(6, 4) \\
6/4 &= 1 \text{ R} \ 2 \Rightarrow \text{gcd}(4, 2) \\
4/2 &= 2 \text{ R} \ 0 \Rightarrow \ 2
\end{align*}
\]
c) \(gcd(912, 625)\)

\[
\begin{align*}
912/625 &= 1 R 287 => gcd(625, 287) \\
625/287 &= 2 R 51 => gcd(287, 51) \\
287/51 &= 5 R 32 => gcd(51, 32) \\
51/32 &= 1 R 19 => gcd(32, 19) \\
32/19 &= 1 R 13 => gcd(19, 13) \\
19/13 &= 1 R 6 => gcd(13, 6) \\
13/6 &= 2 R 1 => gcd(6, 1) \\
\Rightarrow 1
\end{align*}
\]

**Question 3: Rules of Inference**

Use rules of inference to show whether the following arguments are valid or not. Formalize the hypotheses and conclusion, and use the step-by-step method we discussed in class. If an argument is not valid, provide a counterexample.

a) Whenever Hans, the Elephant, sees a mouse, he runs. Whenever Hans runs, the ground is shaking. Whenever the ground shakes, Heidi, the chipmunk, gets angry. Heidi did not get angry. Therefore, Hans did not see a mouse.
b) No UMB professors are Yankees fans. Boris is a Yankees fan. Therefore, Boris is not a UMB professor.

\[\forall x [YF(x) \rightarrow \neg UMBP(x)]\]

\[YF(Boris)\]

\[\therefore \neg UMBP(Boris)\]

(1) \[\forall x [YF(x) \rightarrow \neg UMBP(x)]\] Hyp.
(2) \[YF(Boris) \rightarrow \neg UMBP(Boris)\] R.I. 1
(3) \[YF(Boris)\] Hyp.
(4) \[\neg UMBP(Boris)\] R.I. 2 & 3

The argument is valid.
c) Whenever the Red Sox win, Carl celebrates. Whenever Carl celebrates, he drinks some beer. Carl drinks some beer. Therefore, the Red Sox won.

<table>
<thead>
<tr>
<th>W: The Red Sox win</th>
<th>C: Carl celebrates</th>
<th>B: Carl drinks beer</th>
</tr>
</thead>
<tbody>
<tr>
<td>W -&gt; C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C -&gt; B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>-----------</td>
<td></td>
</tr>
<tr>
<td>W</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This argument is not valid. Proof by counterexample:

If W is false, C is true, and B is true, then all hypotheses are true but the conclusion is false.

**Question 4: Finding Relatively Prime Numbers**

As you know, two numbers are relatively prime if they do not share any divisors greater than 1. Write a function in Java, C, C++, Python, or pseudocode that takes two integers as its input and returns a Boolean value. This value is true if the two inputs are relatively prime and is false otherwise. Write your program into the box on the next page.
procedure relative_prime(a, b: integers)
begin
   d = 2;
   while d <= min(a, b) and (a mod d > 0 or b mod d > 0)
      d := d + 1;
   if d <= min(a, b)
      return false;
   return true;
end
**Question 5: Matrices**

Find a matrix $M$ such that

$$
\begin{bmatrix}
2 & 4 \\
1 & 3
\end{bmatrix}
+ 
\begin{bmatrix}
-6 & 3 \\
0 & -2
\end{bmatrix}
= 
\begin{bmatrix}
4 & 27 \\
6 & 15
\end{bmatrix}
$$

Hint: For each column of $M$ you have to solve a system of linear equations. Write down all of these equations and every step of their solution.

Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ Then we get the following equations:

1. $2m_{11} + 4m_{21} + (-6) = 4 \implies 2m_{11} + 4m_{21} = 10$
2. $m_{11} + 3m_{21} + 0 = 6 \implies m_{11} + 3m_{21} = 6$
3. $2m_{12} + 4m_{22} + 3 = 27 \implies 2m_{12} + 4m_{22} = 24$
4. $m_{12} + 3m_{22} + (-2) = 15 \implies m_{12} + 3m_{22} = 17$

From (2) we get:

(5) $m_{11} = 6 - 3m_{21}$

If we use this to substitute $m_{11}$ in (1) we get:

$12 - 6m_{21} + 4m_{21} = 10 \implies -2m_{21} = -2 \implies m_{21} = 1$

From (5) it follows:

$m_{11} = 6 - 3 \times 1 = 3$

From (4) we get:

(6) $m_{12} = 17 - 3m_{22}$

If we use this to substitute $m_{12}$ in (3) we get:

$34 - 6m_{22} + 4m_{22} = 24 \implies -2m_{22} = -10 \implies m_{22} = 5$

From (6) it follows:

$m_{12} = 17 - 3 \times 5 = 2$

Therefore, the solution is $M = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$
Question 6: Proofs

Prove or disprove the following statements:

a) $2^n + 3$ is prime for all positive integers $n$.

This statement is false. Proof by counterexample:

For $n = 5$, $2^5 + 3 = 35$, which is not prime, since $35 = 5 \cdot 7$

b) The product of three odd integers is odd.

For any odd integers $a$, $b$, $c$ we can write $a = 2p + 1$, $b = 2q + 1$, and $c = 2r + 1$ with integers $p$, $q$, and $r$. Then we have:

$$abc = (2p + 1)(2q + 1)(2r + 1) = 8pqr + 4pq + 4pr + 4qr + 2p + 2q + 2r + 1$$

$$abc = 2(4pqr + 2pq + 2pr + 2qr + p + q + r) + 1$$

Since the expression in parentheses is clearly an integer, the product $abc$ must be odd.
c) For every positive integer $n$, $n(n + 1)$ is even.

We first show that the product of an even number $m = 2k$ and an odd number $n = 2q + 1$ is even:

$$mn = 2k(2q + 1) = 4kq + 2k = 2(2kq + k),$$

where $2kq + k$ is an integer.

Now if $n$ is even, then $n + 1$ is odd, so $n(n + 1)$ is odd.

And if $n$ is odd, then $n + 1$ is even, so $n(n + 1)$ is odd.

There are no other possibilities, so we have proven the statement to be true for all integers $n$.

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**Question 7: Formalization of Logical Expressions**

Write down the following statements using predicate logic. Give your propositional functions intuitive names so that you do not have to explain them.

a) Andreas beats up all professors that fail him and do not fail his sister Susanne.

\[
\forall x [PROF(x) \land FAILS(x, Andreas) \land \neg FAILS(x, Susanne) \rightarrow BEATSUP(Andreas, x)]
\]

b) Franziska and Albert never attend the same classes.

\[
\neg \exists x [CLASS(x) \land ATTENDS(Franziska, x) \land ATTENDS(Albert, x)]
\]
c) There is no computer scientist who can dance and sing.

\[ \neg \exists x [\text{CS}(x) \land \text{CANDANCE}(x) \land \text{CANSING}(x)] \]

**Question 8: Recursion**

Give a recursive definition of each of the following sequences (n = 1, 2, 3, …):

a) \( a_n = n \)

\[
\begin{align*}
    a_1 &= 1 \\
    a_{n+1} &= a_n + 1
\end{align*}
\]

b) \( a_n = 2n^2 \)

\[
\begin{align*}
    a_1 &= 2 \\
    a_{n+1} &= a_n + 4n + 2
\end{align*}
\]

c) \( a_n = 3n! \)

\[
\begin{align*}
    a_1 &= 3 \\
    a_{n+1} &= (n+1) a_n
\end{align*}
\]

**Question 9: Converting Back and Forth**

Show every step of your computation for the following conversions:
a) Convert the decimal number 1738 into its hexadecimal expansion.

\[
\begin{align*}
1738 \div 16 &= 108 \text{ R } 10 \\
108 \div 16 &= 6 \text{ R } 12 \\
6 \div 16 &= 0 \text{ R } 6 \\
\end{align*}
\]

Answer: 6CA

b) Convert the binary number \((1001101)_2\) into its decimal expansion.

\[
(1001101)_2 = 2^6 + 2^3 + 2^2 + 2^0 = 64 + 8 + 4 + 1 = 77
\]
Question 10: Some Induction

Use mathematical induction to show that the following equation is true for all natural numbers \( n \):

\[ 2^0 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1 \]

Basis step: \( n = 0 \)

\[ 2^0 = 2^1 - 1 \]
\[ 1 = 1 \text{ true!} \]

Inductive step: To show: \( P(n) \rightarrow P(n+1) \)

\( P(n): \quad 2^0 + 2^1 + 2^2 + \ldots + 2^n + 2^{(n+1)} = 2^{(n+1)} + 2^{(n+1)} - 1 \)

\[ \Rightarrow \quad 2^0 + 2^1 + 2^2 + \ldots + 2^n + 2^{(n+1)} = 2 \cdot 2^{(n+1)} - 1 \]

\[ \Rightarrow \quad 2^0 + 2^1 + 2^2 + \ldots + 2^n + 2^{(n+1)} = 2^{(n+2)} - 1 \]

This is \( P(n+1) \).

Conclusive step: The statement is true for all natural numbers \( n \).