Question 1:

a) Write down the code of Program $P$ in the language $L$ for which $\#(P) = 38399$.

$38400 = 2^9 \cdot 3^1 \cdot 5^2 = [9, 1, 2]$

This means that this program has three instructions, and their codes are 9, 1, and 2.

First instruction:

$9 = <1, 2> = <1, <0, 1>>$, so $a = 1$, $b = 0$, and $c = 1$.

This means the label is A, the instruction type is $V \leftarrow V$, and the variable is X, so we have:

[A]  X ← X

Second instruction:

$1 = <1, 0> = <1, <0, 0>>$, so $a = 1$, $b = 0$, and $c = 0$.

This means the label is A, the instruction type is $V \leftarrow V$, and the variable is Y, so we have:

[A]  Y ← Y

Third instruction:

$2 = <0, 1> = <0, <1, 0>>$, so $a = 0$, $b = 1$, and $c = 0$.

This means the instruction is unlabeled, the instruction type is $V \leftarrow V + 1$, and the variable is Y, so we have:

Y ← Y + 1
The whole program is:

[A]  X ← X
[A]  Y ← Y
     Y ← Y + 1

According to our specification of the language \( \mathcal{L} \), it is OK to have the same label twice within a program. If we wanted to jump to A, by definition we would jump to the first instruction with that label, so here it would be the first instruction. It does not really make sense to have such a duplicate, but it still a valid \( \mathcal{L} \) program.

b) What is the number of the following program?

\[
\begin{align*}
&\text{IF } X \neq 0 \text{ GOTO C} \\
&Y ← Y - 1 \\
&\text{[C]} \quad Y ← Y + 1
\end{align*}
\]

You do not have to compute the numerical values of expressions such as \( 3^{27} \) that would result in huge numbers.

First instruction:

\[
\begin{align*}
a &= 0, \quad b = 2 + 3 = 5, \quad c = 1 \\
<0, <5, 1>> &= <0, 95> = 190
\end{align*}
\]

Second instruction:

\[
\begin{align*}
a &= 0, \quad b = 2, \quad c = 0 \\
<0, <2, 0>> &= <0, 3> = 6
\end{align*}
\]

Third instruction:

\[
\begin{align*}
a &= 3, \quad b = 1, \quad c = 0 \\
<3, <1, 0>> &= <3, 1> = 23
\end{align*}
\]

So the number of the program is \([190, 6, 23] \} = 2^{190} \cdot 3^6 \cdot 5^{23} - 1

(which, by the way, is
13637564078262121310342223911042180113013437745017651199999999999999999999999)
**Question 2:**

Do you remember how we used the pairing function and the Gödel numbering to associate each program in the language \( L \) with a unique natural number? To be precise, we demanded that every program in \( L \) is associated with a unique number, and we also required that every natural number is associated with a valid program in \( L \). Now it is your task to develop such one-to-one mappings for other things. If you think that a mapping cannot be defined, please give a reason.

a) Define such a mapping for the set of rational numbers \( \mathbb{Q} \).

OK, this is an extremely difficult question and is thus a bonus question. Any reasonable ideas or approaches will give partial credit.

As shown in (b) below, we can define a bijection \( m \) from the natural numbers to the integers. Now rational numbers are fractions, and therefore we can use our pairing function to derive a function \( f(x) \) that maps the natural numbers onto the rational ones:

\[
 f(x) = \frac{m[l(x)]}{r(x) + 1}
\]

This function is defined for all natural numbers and is surjective, i.e., can produce any rational number as its output. The problem is that it is not injective, that is, it maps multiple natural numbers onto the same rational number.

We have to avoid such duplicates by defining a function \( g(x) \) that finds for a given input \( x \) the \((x+1)\)-th unique number in the sequence \( f(0), f(1), f(2), \ldots \):

\[
 g(0) = 0 \\
 g(x + 1) = f[\min,\{|f(0), f(1), \ldots, f(i) | \geq x + 1\}]
\]

It uses (unbounded) minimalization and the cardinality function to determine the desired number. So \( g(x) \) is the mapping we are looking for.

b) Define such a mapping for the set of integers.

The following function \( m: \mathbb{N} \to \mathbb{Z} \) provides such a mapping:
c) Define such a mapping for the set of letters of the English alphabet.

It is impossible to map each member of an infinite set - such as the natural numbers – onto a unique member of a finite set of 26 members. It is obvious that there have to be duplicates. Therefore, we cannot define such a mapping.