Midterm Practice Exam
Sample Solutions

Duration: 1 hour and 15 minutes

You only need your writing utensils to complete this exam. No calculators, no books, and no notes allowed. Use the back of any page for scratch work. There are two blank pages at the end in case you run out of room for an answer.

Question 1: ____ out of ____ points

Question 2: ____ out of ____ points

Question 3: ____ out of ____ points

Question 4: ____ out of ____ points

Total Score:

Grade:
**Question 1: True or False?**

Are the following statements true or false? Check the appropriate box for each statement. Notice that you will get 2 points for every correct answer but lose 1 point for an incorrect one; you can leave both boxes blank if you are not sure which answer is correct.

<table>
<thead>
<tr>
<th>Statement</th>
<th>true</th>
<th>false</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Every partially computable function is also computable.</td>
<td>[ ]</td>
<td>[X]</td>
</tr>
<tr>
<td>b) There are infinitely many programs in the language $L$ that compute the function $f(x) = 2x$.</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>c) The class of primitive recursive functions is a subset of every PRC class.</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>d) If the language $L$ included the instruction “GOTO L” instead of the instruction “IF V≠0 GOTO L,” then $\text{HALT}(x, y)$ were computable.</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>e) $[3, 2, 1] = 360$</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>f) If function $g(x)$ is partially computable and function $h(x)$ is computable, then function $f(x) = g(h(x))$ is partially computable.</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>g) Every function in a PRC class can be derived from the initial functions by applying composition and primitive recursion a finite number of times.</td>
<td>[ ]</td>
<td>[X]</td>
</tr>
<tr>
<td>h) $\langle 3, 2, 1 \rangle = 112$</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>i) The snapshot of a computation in the language $L$ is given by the values of all its variables and the number of the next instruction to be executed.</td>
<td>[X]</td>
<td>[ ]</td>
</tr>
<tr>
<td>j) The function $f(x, y) = x – y$ is computable in the language $L$.</td>
<td>[ ]</td>
<td>[X]</td>
</tr>
</tbody>
</table>
Question 2: Expansive Code

Please take a look at the following program called MYSTERY:

[A]  IF X≠0 GOTO B
    Y ← Y+1
    IF Y≠0 GOTO E

[B]  X ← X-1
    IF X≠0 GOTO C
    Z ← Z+1
    IF Z≠0 GOTO E

[C]  X ← X-1
    Z ← Z+1
    IF Z≠0 GOTO A

(a) What function does this code compute? You can describe it in English words instead of equations if you like.

This function computes the predicate “is_even.”

(b) You now employ a macro MYSTERY(x) to use this program in another, embedding one named ENIGMA(x):

[A₃]  X ← X + 1
      Y ← MYSTERY(X)

Expand the macro MYSTERY(X) and show the resulting, expanded version of ENIGMA(x). You do not have to further expand macros such as V ← 0, V ← V’, or GOTO L. You get full points if your program is correct and bonus points if you perform the expansion exactly as described in the textbook.
[A3] \[ X \leftarrow X + 1 \\
Z_4 \leftarrow 0 \\
Z_5 \leftarrow X \\
Z_6 \leftarrow 0 \]

[A5] \[ \text{IF } Z_5 \neq 0 \text{ GOTO } A_6 \\
Z_4 \leftarrow Z_4 + 1 \\
\text{IF } Z_4 \neq 0 \text{ GOTO } E_4 \]

[A6] \[ Z_5 \leftarrow Z_5 - 1 \\
\text{IF } Z_5 \neq 0 \text{ GOTO } A_7 \\
Z_6 \leftarrow Z_6 + 1 \\
\text{IF } Z_6 \neq 0 \text{ GOTO } E_4 \]

[A7] \[ Z_5 \leftarrow Z_5 - 1 \\
Z_6 \leftarrow Z_6 + 1 \\
\text{IF } Z_6 \neq 0 \text{ GOTO } A_5 \]

[E4] \[ Y \leftarrow Z_4 \]

**Question 3: A New Type of Prime Number**

Dealing with the same old prime numbers has become boring. Let us define a new type of prime number, which we will call square prime. A square prime is a natural number \(n\) greater than 1 that is a perfect square and has no other divisors than 1, \(n\), and the square root of \(n\).

For example, the smallest square prime is 4, because 4 is a perfect square \((2 \cdot 2 = 4)\) and its only divisors are 1, 2, and 4. The number 9 is the next square prime, because we have \(3 \cdot 3 = 9\), and its only divisors are 1, 3, and 9. However, the next perfect square, 16, is not a square prime: Besides the divisors 1, 4, and 16, it is also divisible by 2 and 8. The following one, 25, is a square prime, being only divided by 1, 5, and 25. And so on…

The predicate \(\text{SquarePrime}(x)\) is TRUE, if \(x\) is a square prime, and FALSE otherwise. Show that \(\text{SquarePrime}(x)\) is a primitive recursive predicate. In your proof, you can refer to all functions and predicates that we have already shown in class to be primitive recursive.

One possibility:

\[
\text{SquarePrime}(x) = x > 1 \land (\exists t)(t \cdot t = x) \land (\forall s)(\neg(s \mid x) \lor s \cdot s = x \lor s = 1)
\]

In this equation, \(\text{SquarePrime}\) is defined through composition of other functions that we have previously shown to be primitive recursive. Therefore, \(\text{SquarePrime}\) must also be primitive recursive.
Question 4: About Programs and Instructions

a) In our enumeration scheme for programs in the language $L$, each instruction is completely described by the variables $a$, $b$, and $c$. The variable $c$ is defined as

$$c = #(V) - 1.$$  

Explain why the subtraction of 1 is necessary to make our enumeration scheme work.

b) Write down the instruction $I$ for which $#(I) = 93$. Explain every step of your calculation.

a) If we did not subtract 1, then $c$ could never be 0, because our enumeration of variables starts at number 1. That would be OK for translating programs into numbers; we would still associate every possible program with a unique number. However, there would be numbers that did not translate into programs. This is because some numbers would result in $c = 0$ for one or more instructions, and these instructions would be undefined.

b) $#(I) = <a, <b, c>>$

\[ 93 + 1 = 2^a(2^{<b, c>} + 1) \]  

// resolve first pairing to determine $a$ and $<b, c>$

\[ a = 1 \]

\[ 2^{<b, c>} + 1 = 47 \]

\[ <b, c> = 23 \]

\[ 24 = 2^b(2^c + 1) \]  

// resolve second pairing to determine $b$ and $c$

\[ b = 3 \]

\[ c = 1 \]

$a = 1$ means that the instruction is labeled [A].

$b = 3$ means that the instruction type is a conditional branch to label A.

c = 1 means that the variable in this instruction is X

The instruction then is:

[A] IF X≠0 GOTO A