Numerical Representation of Strings

First, we define two primitive recursive functions

\[ R^+(x, y) = \begin{cases} R(x, y) & \text{if } y \mid x \\ y & \text{otherwise} \end{cases} \]

\[ Q^+(x, y) = \begin{cases} \lfloor x / y \rfloor & \text{if } y \mid x \\ \lfloor x / y \rfloor - 1 & \text{otherwise} \end{cases} \]

where \( R(x, y) \) and \( \lfloor x / y \rfloor \) are defined as in Section 3.7.

Basically, \( R^+ \) and \( Q^+ \) are the “usual” remainder and quotient functions, except that remainders are now in the range between 1 and \( y \) instead of 0 and \( y - 1 \).

So whenever \( y \) divides \( x \), we do not have a remainder of 0 but a remainder of \( y \), and accordingly the quotient is one number below the “actual” quotient.

Therefore, like with the usual quotient and remainder, it is still true that:

\[ x/y = Q^+(x, y) + R^+(x, y)/y, \]

only that now we have \( 1 \leq R^+(x, y) \leq y \).

We will use the functions \( Q^+ \) and \( R^+ \) to show how to obtain the subscripts \( i_0, i_1, \ldots, i_k \) from any integer \( x > 0 \).

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Let us define:

\[ u_0 = x \]
\[ u_{m+1} = Q^+(u_m, n) \]

Then we have:

\[ u_0 = i_k \cdot n^k + i_{k-1} \cdot n^{k-1} + \ldots + i_1 \cdot n^1 + i_0 \]
\[ u_1 = i_k \cdot n^{k-1} + i_{k-1} \cdot n^{k-2} + \ldots + i_1 \]
\[ u_k = i_k \]

The “remainders” \( R^+ \) are exactly the values of the \( i_m \):

\[ i_m = R^+(u_m, n), \ m = 0, \ldots, k. \]

This is analogous to our usual base-\( n \) notation:

\[ u_0 = x \]
\[ u_{m+1} = Q(u_m, n) \]

Then we have:

\[ u_0 = i_k \cdot n^k + i_{k-1} \cdot n^{k-1} + \ldots + i_1 \cdot n^1 + i_0 \]
\[ u_1 = i_k \cdot n^{k-1} + i_{k-1} \cdot n^{k-2} + \ldots + i_1 \]
\[ u_k = i_k \]

The remainders \( R \) are exactly the values of the \( i_m \):

\[ i_m = R(u_m, n), \ m = 0, \ldots, k. \]

Example: Find binary representation of number 13:

Then \( u_0 = x = 13; n = 2 \)

\[ u_1 = Q(13, 2) = 6; \ i_0 = R(13, 2) = 1 \]
\[ u_2 = Q(6, 2) = 3; \ i_1 = R(6, 2) = 0 \]
\[ u_3 = Q(3, 2) = 1; \ i_2 = R(3, 2) = 1 \]
\[ u_4 = Q(1, 2) = 0; \ i_3 = R(1, 2) = 1 \]

Then \( w = 1101 \).

Thus \( k = 3 \) and we have

\[ x = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \]

You certainly noticed that in our string representation we used symbols \( s_1, \ldots, s_{10} \), while in the everyday number representation we use \( s_0, \ldots, s_{10} \).

So what exactly are the analogies and differences between the two systems?

To find out about this, let us look at the following modified set of digits \( D \):

\[ D = \{ s_1, \ldots, s_{10} \} = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, X \} \]

Here, the X stands for a digit with the value 10, while there is no digit with the value 0.
Numerical Representation of Strings

So what is the number x associated with the string w = 76?
\[ x = 7 \cdot 10 + 6 = 76 \]

And what is the number for w = 3X6?
\[ x = 3 \cdot 100 + 10 \cdot 10 + 6 = 406 \]

Finally, what is the number for w = XX?
\[ x = 10 \cdot 10 + 10 = 110 \]

These examples already suggest that we can use this system in a way quite similar to our “usual” system.

Numerical Representation of Strings

Now let us turn this around: What is the string w associated with the number x = 39?
\[ w = 39 \text{ (as long as x does not contain any 0s, w is the “usual” decimal string representing x)} \]

And what is the string for x = 100?
\[ w = 9X \]

But what is the string for x = 504?
\[ w = 4X4 \]

Finally, what is the string for x = 0?
\[ w = 0 \text{ (0 is the null string symbol)} \]

Numerical Representation of Strings

We can even transfer our elementary arithmetic to the new system:

\[
\begin{array}{c}
X 4 \\
+ 5 9 6 \\
\hline
6 9 X
\end{array}
\text{ corresponding to } \begin{array}{c}
1 0 4 \\
+ 5 9 6 \\
\hline
7 0 0
\end{array}
\]

\[
\begin{array}{c}
X 2 3 \\
- X 1 X \\
\hline
3
\end{array}
\text{ corresponding to } \begin{array}{c}
1 0 2 3 \\
- 1 0 2 0 \\
\hline
3
\end{array}
\]

Numerical Representation of Strings

This applies even to multiplication:

\[
\begin{array}{c}
3 X \\
\times X 7 \\
\hline
2 7 X
\end{array}
\text{ corresponding to } \begin{array}{c}
4 0 \\
\times 1 0 7 \\
\hline
4 2 8 0
\end{array}
\]

\[
\begin{array}{c}
3 9 X \\
\hline
4 2 7 X
\end{array}
\]