**Theorem 6.1:** Let $C$ be a PRC class. If $f(t, x_1, \ldots, x_n)$ belongs to $C$, then so do the functions:

$$g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

$$h(y, x_1, \ldots, x_n) = \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

To prove this theorem, we will show that $g$ and $h$ can be obtained from $f$ by primitive recursion.

**Proof:**

To prove this theorem, we will show that $g$ and $h$ can be obtained by recursion in the following way:

$$g(0, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n)$$

$$g(t + 1, x_1, \ldots, x_n) = g(t, x_1, \ldots, x_n) + f(t + 1, x_1, \ldots, x_n)$$

Since $+$ is primitive recursive, $g$ belongs to $C$.

**Proof II:**

$$h(0, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n)$$

$$h(t + 1, x_1, \ldots, x_n) = h(t, x_1, \ldots, x_n) \cdot f(t + 1, x_1, \ldots, x_n)$$

Since $\cdot$ is primitive recursive, $h$ belongs to $C$.

As a “by-product” of the preceding idea we obtained the following corollary:

**Corollary 6.2:** If $f(t, x_1, \ldots, x_n)$ belongs to some PRC class $C$, then so do the following two functions:

$$g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

$$h(y, x_1, \ldots, x_n) = \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

**Theorem 6.3:** If the predicate $P(t, x_1, \ldots, x_n)$ belongs to some PRC class $C$, then so do the following two predicates:

$$(\forall t) \exists y P(t, x_1, \ldots, x_n) \quad \text{and} \quad (\exists t) \exists y P(t, x_1, \ldots, x_n)$$

**Proof:**

$$(\forall t) \exists y P(t, x_1, \ldots, x_n) \iff \left[ \prod_{t=0}^{y} P(t, x_1, \ldots, x_n) \right] = 1$$

$$(\exists t) \exists y P(t, x_1, \ldots, x_n) \iff \left[ \sum_{t=0}^{y} P(t, x_1, \ldots, x_n) \right] \neq 0$$
**Primitive Recursive Predicates**

Sometimes we may want to use the quantifiers 

\((\forall t)_y P(t, x_1, \ldots, x_n)\) and \((\exists t)_y P(t, x_1, \ldots, x_n)\).

Then Theorem 6.3 is still valid, which is obvious from the following two relations:

\[(\forall t)_y P(t, x_1, \ldots, x_n) \Leftrightarrow (\forall t)_y [t = y \lor P(t, x_1, \ldots, x_n)]\]

\[(\exists t)_y P(t, x_1, \ldots, x_n) \Leftrightarrow (\exists t)_y [t = y \land P(t, x_1, \ldots, x_n)]\]

---

**Example 12:** \(y \mid x\)

\(y \mid x\) means "y is a divisor of x."

For example,

- \(4 \mid 15\) is false.
- \(3 \mid 9\) is true.

This predicate is primitive recursive because

\[y \mid x \Leftrightarrow (\exists t)_{\leq y} (y \cdot t = x)\]

---

**Minimalization**

Let the predicate \(P(t, x_1, \ldots, x_n)\) belong to some PRC class \(C\).

Then by Theorem 6.1 the function

\[g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} \prod_{i=0}^{n} \alpha(P(t, x_1, \ldots, x_n))\]

also belongs to \(C\).

(Remember the primitive recursive "negation" function \(\alpha\) we defined earlier.)

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**Example 13:** \(Prime(x)\)

Prime(x) is the predicate "x is a prime."

It is primitive recursive since

\[Prime(x) \Leftrightarrow x > 1 \land (\forall t)_{\leq y} [t = 1 \lor t = x \lor \neg (t \mid x)].\]

---

**Minimalization**

Let us take a closer look at the function

\[g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} \prod_{i=0}^{n} \alpha(P(t, x_1, \ldots, x_n))\]

Let us assume that there is a value \(t_0\) that is the smallest value of \(t \leq y\) for which \(P(t, x_1, \ldots, x_n)\) is true:

\[P(t, x_1, \ldots, x_n) = 0\] for \(t < t_0\)

\[P(t_0, x_1, \ldots, x_n) = 1.\]

Then

\[\prod_{i=0}^{n} \alpha(P(t, x_1, \ldots, x_n)) = \begin{cases} 1 & \text{if } u < t_0 \\ 0 & \text{if } u \geq t_0 \end{cases}\]

Hence,

\[g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} 1 = t_0\]

so that \(g(y, x_1, \ldots, x_n)\) is the least value of \(t\) for which \(P(t, x_1, \ldots, x_n)\) is true.

---

Then we define:

\[
\min_{t \leq y} P(t, x_1, \ldots, x_n) = \begin{cases} g(y, x_1, \ldots, x_n) & \text{if } (\exists t)_{\leq y} P(t, x_1, \ldots, x_n) \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, \(\min_{t \leq y} P(t, x_1, \ldots, x_n)\) is the smallest value of \(t\) for which \(P(t, x_1, \ldots, x_n)\) is true, if such \(t\) exists, otherwise it is 0.

Using Theorems 5.4 and 6.3, we have:

**Theorem 7.1:** If \(P(t, x_1, \ldots, x_n)\) belongs to some PRC Class \(C\) and \(f(y, x_1, \ldots, x_n) = \min_{t \leq y} P(t, x_1, \ldots, x_n)\), then \(f\) also belongs to \(C\).

We will call the operation "mint\(\leq y\)" bounded minimalization.
Minimalization

Example 14: \( \lfloor \frac{x}{y} \rfloor \)
- \( \lfloor \cdot \rfloor \) is the floor function.
- For example, \( \lfloor 8/3 \rfloor = 2 \).
- \( \lfloor x/y \rfloor \) is the integer part of the quotient \( x/y \).
- \( \lfloor x/y \rfloor \) is primitive recursive as shown by the equation
  \[ \lfloor x/y \rfloor = \min_{t \leq x} \{ (t + 1) \cdot y > x \} \]
- Note that this equation gives us \( \lfloor x/0 \rfloor = 0 \).

Minimalization

Example 15: \( R(x, y) \)
- \( R(x, y) \) is the remainder when \( x \) is divided by \( y \).
- We can also write \( R(x, y) = x \mod y \) ("modulo").
- Obviously, it is true that \( x/y = \lfloor x/y \rfloor + R(x, y)/y \)
- Therefore, we can write:
  \[ R(x, y) = x - \sum_{i=0}^{\lfloor x/y \rfloor} (y \cdot \lfloor x/y \rfloor) \]
- This shows that \( R(x, y) \) is primitive recursive.
- Note that \( R(x, 0) = x \).

Minimalization

Example 16: \( p_n \)
- Here, for \( n > 0 \), \( p_n \) is the \( n \)-th prime number (in order of size).
- In order to make \( p_n \) a total function, we set \( p_0 = 0 \).
- Then we have:
  \( p_0 = 0 \)
  \( p_1 = 2 \)
  \( p_2 = 3 \)
  \( p_3 = 5 \)
  \( p_4 = 7 \)

Now consider the following recursion equations:
\[ p_0 = 0 \]
\[ p_{n+1} = \min_{t \geq p_n} \{ \text{Prime}(t) \land t > p_n \} \]
- To see that these equations are correct, we must verify the following inequality:
  \( p_{n+1} \leq p_n! + 1 \).
- Note that for \( 0 < i \leq n \) we have:
  \( (p_i! + 1) / p_i = p_i / p_i + 1/p_i = K + 1/p_i \),
  where \( K \) is an integer.
- Why is \( p_i! \) always divisible by \( p_i \)?
- Example: \( n = 4, i = 2 \)
  Then \( p_2! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \), and \( p_2 = 3 \),
  so \( p_2! / p_2 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = K \).
  In other words, \( p_i \) is always one of the factors in \( p_i! \).
  According to the equation \( (p_i! + 1) / p_i = K + 1/p_i \),
  \( p_i! + 1 \) is not divisible by any of the primes \( p_1, ..., p_{i-1} \).

So either \( p_i! + 1 \) is itself a prime or it is divisible by a prime > \( p_i \).
- In either case there is a prime \( q \) such that
  \( p_n < q \leq p_i! + 1 \), which gives us the inequality that we wanted to verify:
  \( p_{n+1} \leq p_i! + 1 \).